ROMANIAN MATHEMATICAL MAGAZINE

$$\textit{Let the sequence } \{a_n\} \textit{ be given}$$
$$a_1 = 6, a_2 = 14,\ \ a_{n+2} = 5a_{n+1} - 6a_n + 2\ (n \geq 1),$$
$$a_{2024} \equiv x\ (mod\ 100).\ \text{Find } x.$$

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Solution by Abbaszade Yusif-Azerbaijan

$$Let:\ a_n = x^n,\ \ x^{n+2} = 5x^{n+1} - 6x^n$$

$$x^n(x^2 - 5x + 6) = 0,\ \ x_1 = 2\ \ x_2 = 3$$
$$a_n = c_1 2^n + c_2 3^n + c_3$$

$$Let\ a_n = y\ =>\ \ y - 5y + 6y = 2\ =>\ y = 1 = c_1\ =>\ \ a_n = c_1 2^n + c_2 3^n + 1$$

$$6 = 2c_1 + 3c_2 + 1\ =>\ \ 4c_1 + 6c_2 = 10\ =>\ c_1 = 1$$
$$14 = 4c_1 + 9c_2 + 1\ =>\ \ 4c_1 + 9c_2 = 13\ =>\ c_2 = 1$$

$$a_n = 2^n + 3^n + 1\ =>\ \ a_{2024} = 2^{2024} + 3^{2024} + 1$$

$$2^{2024} + 3^{2024} + 1 \equiv x\ (mod\ 100) =>$$
$$2^{24} \times (2^{40})^{50} + 3^{24}\ \times (3^{40})^{50} + 1 \equiv\ x\ (mod\ 100)$$
$$2^{24} + 3^{24} + 1 \equiv\ x\ (mod\ 100)$$
$$x = \left\{\frac{5 \times 4 + 1}{10}\right\} \times 10 + 6 + \left\{\frac{8 - 5 \times 2}{10}\right\} \times 10 + 1 + 1 = 98$$