ROMANIAN MATHEMATICAL MAGAZINE

Prove that there is no positive integers n and x such that

$$\sum_{k=1}^{x^n} k\left(\left\lfloor\frac{2x^n-k}{k}\right\rfloor+\left\lfloor\frac{2x^n-2-k}{2k}\right\rfloor-\left\lfloor\frac{2x^n-k}{2k}\right\rfloor-\left\lfloor\frac{2x^n-2-k}{k}\right\rfloor\right)=x^n+1,$$

where x is a even number, $x>2$ and $\lfloor.\rfloor$ denote the floor function

Proposed by Toubal Fethi-Algeria

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let $a_k=\left\lfloor\frac{2x^n-k}{k}\right\rfloor+\left\lfloor\frac{2x^n-2-k}{2k}\right\rfloor-\left\lfloor\frac{2x^n-k}{2k}\right\rfloor-\left\lfloor\frac{2x^n-2-k}{k}\right\rfloor,\ \ k\ge 1.$

By Hermite's identity, we have $\lfloor 2t\rfloor=\lfloor t\rfloor+\left\lfloor t+\frac{1}{2}\right\rfloor,\ \ \forall t\in\mathbb{R}.$

By using this identity, we have

$$a_k=\left(\left\lfloor 2.\frac{x^n}{k}\right\rfloor-1\right)+\left(\left\lfloor\frac{x^n-1}{k}+\frac{1}{2}\right\rfloor-1\right)-\left(\left\lfloor\frac{x^n}{k}+\frac{1}{2}\right\rfloor-1\right)-\left(\left\lfloor 2.\frac{x^n-1}{k}\right\rfloor-1\right)$$

$$=\left(\left\lfloor\frac{x^n}{k}\right\rfloor+\left\lfloor\frac{x^n}{k}+\frac{1}{2}\right\rfloor\right)+\left\lfloor\frac{x^n-1}{k}+\frac{1}{2}\right\rfloor-\left\lfloor\frac{x^n}{k}+\frac{1}{2}\right\rfloor-\left(\left\lfloor\frac{x^n-1}{k}\right\rfloor+\left\lfloor\frac{x^n-1}{k}+\frac{1}{2}\right\rfloor\right)=$$

$$=\left\lfloor\frac{x^n}{k}\right\rfloor-\left\lfloor\frac{x^n-1}{k}\right\rfloor$$

then $a_1=x^n-(x^n-1)=1,\ \ a_2=\frac{x^n}{2}-\left(\frac{x^n}{2}+\left\lfloor-\frac{1}{2}\right\rfloor\right)=1,\ \ a_{x^n}=1-0=1,$
and $a_k\ge 0,\ \ \forall k\in[\![3,x^n-1]\!]$, therefore

$$\sum_{k=1}^{x^n} ka_k\ge a_1+2a_2+x^na_{x^n}>x^n+1,$$

So there is no positive integers n and x which verify the given equation.