ROMANIAN MATHEMATICAL MAGAZINE

Let n and m be two positive integers such that m ≤ n. Prove that

$$\frac{n-m+2\sum_{k=1}^{n}\left(\left\lfloor\frac{2km-n}{n}\right\rfloor-\left\lfloor\frac{2km-n}{2n}\right\rfloor\right)}{1+lcm(n,m)}$$

is integer, where lcm is the least common multiple and $\lfloor . \rfloor$ denote the floor function.

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By Hermite's identity, we have $\lfloor 2t\rfloor = \lfloor t\rfloor + \left\lfloor t+\frac{1}{2}\right\rfloor, \ \forall t\in\mathbb{R}$, then

$$\left\lfloor\frac{2km-n}{n}\right\rfloor-\left\lfloor\frac{2km-n}{2n}\right\rfloor=\left(\left\lfloor 2.\frac{km}{n}\right\rfloor-1\right)-\left(\left\lfloor\frac{km}{n}+\frac{1}{2}\right\rfloor-1\right)=\left\lfloor\frac{km}{n}\right\rfloor.$$

Now, we will prove the following identity that

$$S=\sum_{k=1}^{n-1}\left\lfloor\frac{km}{n}\right\rfloor=\frac{(m-1)(n-1)+gcd(n,m)-1}{2}.$$

We have

$$2S=\sum_{k=1}^{n-1}\left(\left\lfloor\frac{km}{n}\right\rfloor+\left\lfloor\frac{(n-k)m}{n}\right\rfloor\right)=\sum_{k=1}^{n-1}\left(\left\lfloor\frac{km}{n}\right\rfloor+\left\lfloor-\frac{km}{n}\right\rfloor\right)+(n-1)m.$$

We know that if $x\in\mathbb{Z}$, we have $\lfloor x\rfloor+\lfloor -x\rfloor=0$, otherwise $\lfloor x\rfloor+\lfloor -x\rfloor=-1$.

Let $m=da, n=db$ where $d=gcd(n,m)$ and $a,b\in\mathbb{N}$. The number $\frac{km}{n}$

$$=\frac{ka}{b} \text{ is integer } d-1$$

times for $k=b,2b,\dots,(d-1)b$, then

$$2S=-(n-d)+(n-1)m=(m-1)(n-1)+gcd(n,m)-1,$$

which completes the proof of the identity. Using this identity we have

$$n-m+2\sum_{k=1}^{n}\left(\left\lfloor\frac{2km-n}{n}\right\rfloor-\left\lfloor\frac{2km-n}{2n}\right\rfloor\right)=n-m+2\sum_{k=1}^{n}\left\lfloor\frac{km}{n}\right\rfloor=n-m+2(S+m)$$

$$=n+m+(m-1)(n-1)+gcd(n,m)-1=mn+gcd(n,m)$$

$$=(lcm(n,m)+1).gcd(n,m).$$

$$\Rightarrow \frac{n-m+2\sum_{k=1}^{n}\left(\left\lfloor\frac{2km-n}{n}\right\rfloor-\left\lfloor\frac{2km-n}{2n}\right\rfloor\right)}{1+lcm(n,m)}=gcd(n,m) \text{ is integer.}$$