ROMANIAN MATHEMATICAL MAGAZINE

If $a, b > 0$ with $ab = 1$ and $0 \le \lambda \le 10$, then :

$$\frac{1}{a^3(b+\lambda)} + \frac{1}{b^3(a+\lambda)} + \frac{1}{a+b} \ge \frac{\lambda+5}{2(\lambda+1)}$$

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Solution by Soumava Chakraborty-Kolkata-India

$$\frac{1}{a^3(b+\lambda)} + \frac{1}{b^3(a+\lambda)} + \frac{1}{a+b} \overset{ab=1}{=} \frac{b^3}{b+\lambda} + \frac{a^3}{a+\lambda} + \frac{1}{a+b} \overset{ab=1}{=}$$

$$= \frac{b^2}{1+\lambda a} + \frac{a^2}{1+\lambda b} + \frac{1}{a+b} \overset{ab=1}{=} \frac{\lambda\left((a+b)^3 - 3(a+b)\right) + (a+b)^2 - 2}{1 + \lambda(a+b) + \lambda^2} + \frac{1}{a+b}$$

$$= \frac{\lambda(t^4 - 3t^2) + t^3 - 2t + 1 + \lambda^2 + \lambda t}{1 + \lambda^2 + \lambda t} \ (t = a+b) \boxed{\overset{?}{\ge}} \frac{\lambda+5}{2(\lambda+1)}$$

$$\Leftrightarrow -\lambda^3(t-2) + \lambda^2(2t^4 - 7t^2 - 3t + 2) + \lambda(2t^4 + 2t^3 - 11t^2 - 3t + 2) + 2t^3 - 9t + 2 \ge$$
$$\ge -10\lambda^2(t-2) + \lambda^2(t-2)(2t^3 + 4t^2 + t - 1) + \lambda(t-2)(2t^3 + 6t^2 + t - 1)$$
$$+(t-2)(2t^2 + 4t - 1) \left(\because t = a+b \overset{A-G}{\ge} 2\sqrt{ab} \overset{ab=1}{=} 2 \text{ and } -\lambda^3 \overset{\lambda \le 10}{\ge} -10\lambda^2\right) =$$

$$= (t-2)(\lambda^2(2t^3 + 4t^2 + t - 11) + \lambda(2t^3 + 6t^2 + t - 1) + 2t^2 + 4t - 1)$$

$$= (t-2)\begin{pmatrix} \lambda^2(2t^3 + 4(t^2-4) + t + 5) + \lambda(2t^3 + 6t^2 + (t-2) + 1) \\ +2t^2 + 3t + (t-2) + 1 \end{pmatrix} \overset{\substack{t \ge 2 \\ \text{and} \\ \lambda \ge 0}}{\ge} 0$$

$$\therefore \frac{1}{a^3(b+\lambda)} + \frac{1}{b^3(a+\lambda)} + \frac{1}{a+b} \ge \frac{\lambda+5}{2(\lambda+1)}$$

$$\forall\, a, b > 0 \,\middle|\, ab = 1 \text{ and } 0 \le \lambda \le 10, '' ='' \text{ iff } a = b = 1 \text{ (QED)}$$