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If $a,\mathrm{b}>0$ and $\mathrm{n}\in\mathbb{N}$, then :

$$\frac{1}{a^{\mathrm{n}}}+\frac{1}{\mathrm{b}^{\mathrm{n}}}+\frac{8\mathrm{n}(a+\mathrm{b})}{a+\mathrm{b}+2}\geq 2(2\mathrm{n}+1)$$

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If $\mathrm{n}=0$, LHS − RHS $=0\Rightarrow$ LHS = RHS and so, we now focus on $\mathrm{n}\in\mathbb{N}^*$

and $\because \mathrm{n}\geq 1 \therefore$ via Bernoulli, $\frac{1}{a^{\mathrm{n}}}+\frac{1}{\mathrm{b}^{\mathrm{n}}}+\frac{8\mathrm{n}(a+\mathrm{b})}{a+\mathrm{b}+2}$

$$=\left(1+\frac{1}{a}-1\right)^{\mathrm{n}}+\left(1+\frac{1}{\mathrm{b}}-1\right)^{\mathrm{n}}+\frac{8\mathrm{n}(a+\mathrm{b})}{a+\mathrm{b}+2}\geq$$

$$2+\mathrm{n}\left(\frac{1}{a}-1+\frac{1}{\mathrm{b}}-1\right)+\frac{8\mathrm{n}(a+\mathrm{b})}{a+\mathrm{b}+2}\overset{?}{\geq}2(2\mathrm{n}+1)\Leftrightarrow\frac{a+\mathrm{b}-2a\mathrm{b}}{a\mathrm{b}}+\frac{8(a+\mathrm{b})}{a+\mathrm{b}+2}\overset{?}{\geq}4$$

$$\Leftrightarrow (a+\mathrm{b})^2-2a\mathrm{b}(a+\mathrm{b})+2(a+\mathrm{b})-4a\mathrm{b}+8a\mathrm{b}(a+\mathrm{b})\overset{?}{\geq}4a\mathrm{b}(a+\mathrm{b})+8a\mathrm{b}$$

$$\Leftrightarrow (a+\mathrm{b})^2+2(a+\mathrm{b})+2a\mathrm{b}(a+\mathrm{b}-6)\ \boxed{\underset{(*)}{\overset{?}{\geq}}}\ 0$$

$\boxed{\text{Case 1}}$ $a+\mathrm{b}-6\geq 0$ and then : LHS of $(*)\geq(a+\mathrm{b})^2+2(a+\mathrm{b})>0$
$\Rightarrow (*)$ is true (strict inequality)

$\boxed{\text{Case 2}}$ $a+\mathrm{b}-6<0$ and then : LHS of $(*)\geq(a+\mathrm{b})^2+2(a+\mathrm{b})+$
$\frac{(a+\mathrm{b})^2}{2}.(a+\mathrm{b}-6)\left(\because 2a\mathrm{b}\overset{\text{A-G}}{\leq}\frac{(a+\mathrm{b})^2}{2}\right)\overset{?}{\geq}0$

$$\Leftrightarrow 2\mathrm{t}+4+\mathrm{t}(\mathrm{t}-6)\overset{?}{\geq}0\Leftrightarrow \mathrm{t}^2-4\mathrm{t}+4\overset{?}{\geq}0\Leftrightarrow(\mathrm{t}-2)^2\overset{?}{\geq}0\rightarrow\text{true}\Rightarrow(*)\text{ is true}$$

$\therefore$ combining both cases, $(*)$ is true $\forall\, a,\mathrm{b}>0 \therefore \frac{1}{a^{\mathrm{n}}}+\frac{1}{\mathrm{b}^{\mathrm{n}}}+\frac{8\mathrm{n}(a+\mathrm{b})}{a+\mathrm{b}+2}\geq 2(2\mathrm{n}+1)$

$\forall\, a,\mathrm{b}>0$ and $\mathrm{n}\in\mathbb{N}$, " = " iff $\mathrm{n}=0$ or $a=\mathrm{b}$ (QED)