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If $a,\mathrm{b},\mathrm{c}>0, a^3+\mathrm{b}^3+\mathrm{c}^3=3$ and $\lambda\geq 0$ with $\mathrm{n}\in\mathbb{N}$, then :

$$\sum_{\mathrm{cyc}}\frac{a^{3\mathrm{n}+2}}{\mathrm{b}^2+\lambda}\geq\frac{3}{\lambda+1}$$

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$$\text{Firstly, } a^3+2\mathrm{b}^3\overset{A-G}{\geq}3a\mathrm{b}^2, \mathrm{b}^3+2\mathrm{c}^3\overset{A-G}{\geq}3\mathrm{bc}^2 \text{ and } \mathrm{c}^3+2a^3\overset{A-G}{\geq}3\mathrm{c}a^2$$

$$\therefore 3\sum_{\mathrm{cyc}}a^3\geq 3\sum_{\mathrm{cyc}}a\mathrm{b}^2\Rightarrow\sum_{\mathrm{cyc}}a\mathrm{b}^2\leq\sum_{\mathrm{cyc}}a^3\to(1)$$

$$\text{Now, }\sum_{\mathrm{cyc}}\frac{a^{3\mathrm{n}+2}}{\mathrm{b}^2+\lambda}=\sum_{\mathrm{cyc}}\frac{a^{3\mathrm{n}+3}}{a\mathrm{b}^2+\lambda a}=\sum_{\mathrm{cyc}}\frac{(a^3)^{\mathrm{n}+1}}{a\mathrm{b}^2+\lambda a}\overset{\text{Holder}}{\geq}\frac{\left(\sum_{\mathrm{cyc}}a^3\right)^{\mathrm{n}+1}}{3^{\mathrm{n}-1}\left(\sum_{\mathrm{cyc}}a\mathrm{b}^2+\lambda\sum_{\mathrm{cyc}}a\right)}\geq$$

$$\overset{\text{via (1)}}{\geq}\frac{\left(\sum_{\mathrm{cyc}}a^3\right)^{\mathrm{n}+1}}{3^{\mathrm{n}-1}\left(\sum_{\mathrm{cyc}}a^3+3\lambda\right)}\left(\because 3=\sum_{\mathrm{cyc}}a^3\overset{\text{Holder}}{\geq}\frac{1}{9}\left(\sum_{\mathrm{cyc}}a\right)^3\Rightarrow\sum_{\mathrm{cyc}}a\leq 3\right)=$$

$$\overset{a^3+\mathrm{b}^3+\mathrm{c}^3=3}{=}\frac{(3)^{\mathrm{n}+1}}{3^{\mathrm{n}-1}(3+3\lambda)}\therefore\sum_{\mathrm{cyc}}\frac{a^{3\mathrm{n}+2}}{\mathrm{b}^2+\lambda}\geq\frac{3}{\lambda+1}\ \forall\, a,\mathrm{b},\mathrm{c}>0\,\middle|\,a^3+\mathrm{b}^3+\mathrm{c}^3=3$$

$$\text{and } \lambda\geq 0 \text{ with } \mathrm{n}\in\mathbb{N}, \text{"="} \text{ iff } a=\mathrm{b}=\mathrm{c}=1\ (\text{QED})$$