ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0$, $a+b+c=abc,\lambda\geq 0$ then:

$$\frac{1}{\lambda+ab}+\frac{1}{\lambda+bc}+\frac{1}{\lambda+ca}\leq\frac{3}{\lambda+3}$$

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Solution by Tapas Das-India

$$Let\ a=\frac{1}{x},b=\frac{1}{y},c=\frac{1}{z}\ then\ a+b+c=abc\ or$$

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{xyz}\ or,xy+yz+zx=1\ (1)$$

$$\frac{1}{\lambda+ab}+\frac{1}{\lambda+bc}+\frac{1}{\lambda+ca}=\sum\frac{1}{\lambda+ab}=\sum\frac{1}{\lambda+\frac{1}{x}\cdot\frac{1}{y}}=$$

$$=\sum\frac{xy}{\lambda xy+1}=\frac{1}{\lambda}\sum\left(1-\frac{1}{\lambda xy+1}\right)=$$

$$=\frac{3}{\lambda}-\frac{1}{\lambda}\sum\frac{1^2}{\lambda xy+1}\overset{Bergstrom}{\leq}\frac{3}{\lambda}-\frac{1}{\lambda}\frac{(1+1+1)^2}{\lambda(xy+yz+zx)+3}\overset{(1)}{=}$$

$$=\frac{3}{\lambda}-\frac{1}{\lambda}\frac{9}{\lambda+3}=\frac{1}{\lambda}\left(3-\frac{9}{\lambda+3}\right)=\frac{1}{\lambda}\frac{3\lambda}{\lambda+3}=\frac{3}{\lambda+3}$$

$$Equality\ holds\ for\ a=b=c=\sqrt{3}$$