ROMANIAN MATHEMATICAL MAGAZINE

If $x, y, z > 0, x + y + z = 1$, then :

$$\prod_{cyc}(x+2y) \le \frac{2}{3} + 3\sum_{cyc} xy^2$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\prod_{cyc}(x+2y) \le \frac{2}{3} + 3\sum_{cyc} xy^2 \overset{\because x+y+z=1}{\Leftrightarrow}$$

$$3\left(2\sum_{cyc} x^2y + 4\sum_{cyc} xy^2 + 9xyz\right) \le 2\left(\sum_{cyc} x\right)^3 + 9\sum_{cyc} xy^2$$

$$\Leftrightarrow 3\left(2\sum_{cyc} x^2y + 4\sum_{cyc} xy^2 + 9xyz\right) \le$$

$$2\left(\sum_{cyc} x^3 + 3\left(2xyz + \sum_{cyc} x^2y + \sum_{cyc} xy^2\right)\right) + 9\sum_{cyc} xy^2$$

$$\Leftrightarrow 2\sum_{cyc} x^3 + 3\sum_{cyc} xy^2 \ge 15xyz \to \text{true} \because \sum_{cyc} x^3 \overset{A-G}{\ge} 3xyz \text{ and } \sum_{cyc} xy^2 \overset{A-G}{\ge} 3xyz$$

$$\therefore \prod_{cyc}(x+2y) \le \frac{2}{3} + 3\sum_{cyc} xy^2 \ \forall\, x, y, z > 0 \,|\, x + y + z = 1,$$

$$'' ='' \text{ iff } x = y = z = \frac{1}{3} \text{ (QED)}$$