ROMANIAN MATHEMATICAL MAGAZINE

If $a, b \geq 1$ and $\lambda \geq 2$, then :

$$\lambda\left(\frac{a^4}{b^2+1}+\frac{b^4}{a^2+1}\right)+(\lambda+1)\left(\frac{1}{a}+\frac{1}{b}\right) \geq 3\lambda+2$$

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$$\frac{a^4}{b^2+1}+\frac{b^4}{a^2+1}=\frac{a^4-b^4+b^4}{b^2+1}+\frac{b^4-a^4+a^4}{a^2+1}$$

$$=(a^4-b^4)\left(\frac{1}{b^2+1}-\frac{1}{a^2+1}\right)+\frac{b^4-1+1}{b^2+1}+\frac{a^4-1+1}{a^2+1}$$

$$=\frac{(a^4-b^4)(a^2-b^2)}{(a^2+1)(b^2+1)}+b^2-1+\frac{1}{b^2+1}+a^2-1+\frac{1}{a^2+1}$$

$$=\frac{(a^2+b^2)(a^2-b^2)^2}{(a^2+1)(b^2+1)}+a^2-1+b^2-1+\frac{1}{a^2+1}+\frac{1}{b^2+1}$$

$$\Rightarrow \frac{a^4}{b^2+1}+\frac{b^4}{a^2+1}-1=$$

$$\frac{(a^2+b^2)(a^2-b^2)^2}{(a^2+1)(b^2+1)}+a^2-1+b^2-1-\left(\frac{1}{2}-\frac{1}{a^2+1}\right)-\left(\frac{1}{2}-\frac{1}{b^2+1}\right)$$

$$=\frac{(a^2+b^2)(a^2-b^2)^2}{(a^2+1)(b^2+1)}+a^2-1+b^2-1-\frac{a^2-1}{2(a^2+1)}-\frac{b^2-1}{2(b^2+1)}$$

$$=\frac{(a^2+b^2)(a^2-b^2)^2}{(a^2+1)(b^2+1)}+\frac{(a^2-1)(2a^2+1)}{2(a^2+1)}+\frac{(b^2-1)(2b^2+1)}{2(b^2+1)}$$

$$\therefore \boxed{\lambda\left(\frac{a^4}{b^2+1}+\frac{b^4}{a^2+1}\right)-\lambda}=\lambda.\frac{(a^2+b^2)(a^2-b^2)^2}{(a^2+1)(b^2+1)}+$$

$$\lambda.\left(\frac{(a^2-1)(2a^2+1)}{2(a^2+1)}+\frac{(b^2-1)(2b^2+1)}{2(b^2+1)}\right)$$

$$\boxed{\overset{\lambda\geq 2>0}{\geq} \lambda.\left(\frac{(a^2-1)(2a^2+1)}{2(a^2+1)}+\frac{(b^2-1)(2b^2+1)}{2(b^2+1)}\right)}$$

$$\Rightarrow \lambda\left(\frac{a^4}{b^2+1}+\frac{b^4}{a^2+1}\right)+(\lambda+1)\left(\frac{1}{a}+\frac{1}{b}\right)-(3\lambda+2)$$

$$=\lambda\left(\frac{a^4}{b^2+1}+\frac{b^4}{a^2+1}\right)-\lambda-(\lambda+1)\left(2-\frac{1}{a}-\frac{1}{b}\right)\geq$$

$$\lambda.\left(\frac{(a^2-1)(2a^2+1)}{2(a^2+1)}+\frac{(b^2-1)(2b^2+1)}{2(b^2+1)}\right)-(\lambda+1)\left(\frac{a-1}{a}+\frac{b-1}{b}\right)$$

$$\geq\lambda.\left(\frac{(a^2-1)(2a^2+1)}{2(a^2+1)}+\frac{(b^2-1)(2b^2+1)}{2(b^2+1)}\right)-\left(\lambda+\frac{\lambda}{2}\right)\left(\frac{a-1}{a}+\frac{b-1}{b}\right)$$

$$\left(\because 1\leq\frac{\lambda}{2}\text{ and } a,b\geq 1\Rightarrow\frac{a-1}{a}+\frac{b-1}{b}\geq 0\right)$$

$$=\frac{\lambda(a-1)}{2}\left(\frac{(2a^2+1)(a+1)}{a^2+1}-\frac{3}{a}\right)+\frac{\lambda(b-1)}{2}\left(\frac{(2b^2+1)(b+1)}{b^2+1}-\frac{3}{b}\right)$$

$$=\frac{\lambda(a-1)(2a^4+2a^3-2a^2+a-3)}{2a(a^2+1)}+\frac{\lambda(b-1)(2b^4+2b^3-2b^2+b-3)}{2b(b^2+1)}$$

$$=\frac{\lambda(a-1)^2(2a^3+4a^2+2a+3)}{2a(a^2+1)}+\frac{\lambda(b-1)^2(2b^3+4b^2+2b+3)}{2b(b^2+1)}\geq 0$$

$$\therefore\lambda\left(\frac{a^4}{b^2+1}+\frac{b^4}{a^2+1}\right)-\lambda-(\lambda+1)\left(2-\frac{1}{a}-\frac{1}{b}\right)\geq 0$$

$$\Rightarrow\lambda\left(\frac{a^4}{b^2+1}+\frac{b^4}{a^2+1}\right)+(\lambda+1)\left(\frac{1}{a}+\frac{1}{b}\right)\geq 3\lambda+2$$

$\forall\, a,b\geq 1$ and $\lambda\geq 2$, $''=''$ iff $a=b=1$ (QED)