If $a,b,c>0$ and $n\in\mathbb{N}^*$, then :

$$\frac{a}{a+b}+\frac{b}{b+c}+\frac{c}{c+a}+\frac{1}{n^2}\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)\ge\left(1+\frac{1}{n}\right)^2$$

Proposed by Marin Chirciu-Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\frac{a}{a+b}+\frac{b}{b+c}+\frac{c}{c+a}+\frac{1}{n^2}\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)\ge\left(1+\frac{1}{n}\right)^2$$

$$\Leftrightarrow\sum_{cyc}\frac{a}{a+b}+\frac{1}{n^2}\left(\sum_{cyc}\frac{b}{a}-1+1\right)\ge 1+\frac{2}{n}+\frac{1}{n^2}$$

$$\Leftrightarrow\left(\sum_{cyc}\frac{a}{a+b}-1\right)+\frac{1}{n^2}\left(\sum_{cyc}\frac{b}{a}-1\right)\overset{(*)}{\ge}\frac{2}{n}$$

$$\text{Now, }\left(\sum_{cyc}\frac{a}{a+b}-1\right)+\frac{1}{n^2}\left(\sum_{cyc}\frac{b}{a}-1\right)\overset{A-G}{\ge}\frac{2}{n}\cdot\sqrt{\left(\sum_{cyc}\frac{a}{a+b}-1\right)\left(\sum_{cyc}\frac{b}{a}-1\right)}$$

$$=\frac{2}{n}\cdot\sqrt{\left(\sum_{cyc}\frac{1}{1+x}-1\right)\left(\sum_{cyc}x-1\right)}\left(\text{where }\frac{b}{a}=x,\frac{c}{b}=y,\frac{a}{c}=z\right)\overset{?}{\ge}\frac{2}{n}$$

$$\Leftrightarrow\left(\frac{1}{1+x}+\frac{1}{1+y}+\frac{1}{1+z}-1\right)(x+y+z-1)\overset{?}{\ge}1$$

$$\Leftrightarrow\sum_{cyc}x^2+\sum_{cyc}xy\overset{?}{\ge}xyz\sum_{cyc}x+3\Leftrightarrow\sum_{cyc}x^2+\sum_{cyc}xy\underset{(**)}{\overset{?}{\ge}}\sum_{cyc}x+3\ (\because xyz=1)$$

$$\text{Now, }\sum_{cyc}x^2\ge\frac{1}{3}\left(\sum_{cyc}x\right)^2=\frac{1}{3}\left(\sum_{cyc}\frac{b}{a}\right)\left(\sum_{cyc}x\right)\overset{A-G}{\ge}\frac{1}{3}.3.\sum_{cyc}x\Rightarrow\sum_{cyc}x^2\ge\sum_{cyc}x$$

$$\to(1)\text{ and }\sum_{cyc}xy\overset{A-G}{\ge}3.\sqrt[3]{(xyz)^2}=3\ (\because xyz=1)\to(2)\therefore(1)+(2)$$

$$\Rightarrow(**)\Rightarrow(*)\text{ is true}\therefore\frac{a}{a+b}+\frac{b}{b+c}+\frac{c}{c+a}+\frac{1}{n^2}\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)\ge\left(1+\frac{1}{n}\right)^2$$

$$\forall a,b,c>0\text{ and }n\in\mathbb{N}^*,''=''\text{ if }a=b=c\text{ and if}$$

$$\left(\sum_{cyc}\frac{a}{a+b}-1\right)=\frac{1}{n^2}\left(\sum_{cyc}\frac{b}{a}-1\right)\Rightarrow\text{if }n=2$$

$$\text{i.e.},''=''\text{ iff }(a=b=c\text{ and }n=2)\ (\text{QED})$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let $x := a^2 + b^2 + c^2$ and $y := ab + bc + ca$. By CBS inequality, we have

$$\frac{a}{a+b} + \frac{b}{b+c} + \frac{c}{c+a} \geq \frac{(a+b+c)^2}{\sum_{cyc} a(a+b)} = \frac{x+2y}{x+y} = 1 + \frac{y}{x+y}.$$

$$\frac{b}{a} + \frac{c}{b} + \frac{a}{c} \geq \frac{(b+c+a)^2}{ba+cb+ac} = \frac{x+2y}{y} = \frac{x+y}{y} + 1.$$

Using these inequalities, we have

$$\frac{a}{a+b} + \frac{b}{b+c} + \frac{c}{c+a} + \frac{1}{n^2}\left(\frac{b}{a} + \frac{c}{b} + \frac{a}{c}\right) \geq 1 + \left(\frac{y}{x+y} + \frac{x+y}{n^2 y}\right) + \frac{1}{n^2}$$

$$\overset{AM-GM}{\geq} 1 + \frac{2}{n} + \frac{1}{n^2} = \left(1 + \frac{1}{n}\right)^2.$$

Equality holds iff $a = b = c$ and $\frac{y}{x+y} = \frac{x+y}{n^2 y} \Leftrightarrow a = b = c$ and $n = 2$.