ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0, a+b+c=3$ and $\lambda\ge\frac{3}{5}$, then :

$$\sum_{cyc}\frac{a^3}{a^2+\lambda b}\ge\frac{3}{\lambda+1}$$

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$$\sum_{cyc}\frac{a^3}{a^2+\lambda b}=\sum_{cyc}\frac{a^4}{a^3+\lambda ab}\overset{\text{Bergstrom}}{\ge}\frac{\left(\sum_{cyc}a^2\right)^2}{\sum_{cyc}a^3+\lambda\sum_{cyc}ab}\overset{a+b+c=3}{=}$$

$$\frac{\left(\sum_{cyc}a^2\right)^2}{\frac{(\sum_{cyc}a^3)(\sum_{cyc}a)}{3}+\frac{\lambda(\sum_{cyc}ab)(\sum_{cyc}a)^2}{9}}=\frac{9\left(\sum_{cyc}a^2\right)^2}{3(\sum_{cyc}a^3)(\sum_{cyc}a)+\lambda(\sum_{cyc}ab)(\sum_{cyc}a)^2}$$

$$\overset{?}{\ge}\frac{3}{\lambda+1}\Leftrightarrow 3\lambda\left(\sum_{cyc}a^2\right)^2+3\left(\sum_{cyc}a^2\right)^2\overset{?}{\ge}3\left(\sum_{cyc}a^3\right)\left(\sum_{cyc}a\right)$$

$$+\lambda\left(\sum_{cyc}ab\right)\left(\sum_{cyc}a\right)^2\Leftrightarrow$$

$$\lambda\left(3\left(\sum_{cyc}a^2\right)^2-\left(\sum_{cyc}ab\right)\left(\sum_{cyc}a\right)^2\right)+3\left(\sum_{cyc}a^2\right)^2\overset{?}{\underset{(*)}{\ge}}3\left(\sum_{cyc}a^3\right)\left(\sum_{cyc}a\right)$$

$$\text{Now, }3\left(\sum_{cyc}a^2\right)^2-\left(\sum_{cyc}ab\right)\left(\sum_{cyc}a\right)^2\ge$$

$$3\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}ab\right)-\left(\sum_{cyc}ab\right)\left(\sum_{cyc}a\right)^2=\left(\sum_{cyc}ab\right)\left(3\sum_{cyc}a^2-\left(\sum_{cyc}a\right)^2\right)$$

$$\ge 0\text{ and }\because\lambda\ge\frac{3}{5}\therefore\text{LHS of (1)}\ge$$

$$\frac{3}{5}\cdot\left(3\left(\sum_{cyc}a^2\right)^2-\left(\sum_{cyc}ab\right)\left(\sum_{cyc}a\right)^2\right)+3\left(\sum_{cyc}a^2\right)^2\overset{?}{\ge}3\left(\sum_{cyc}a^3\right)\left(\sum_{cyc}a\right)$$

$$\Leftrightarrow 8\left(\sum_{cyc}a^2\right)^2\overset{?}{\underset{(**)}{\ge}}\left(\sum_{cyc}ab\right)\left(\sum_{cyc}a\right)^2+5\left(\sum_{cyc}a^3\right)\left(\sum_{cyc}a\right)$$

Assigning $b+c=x, c+a=y, a+b=z\Rightarrow x+y-z=2c>0, y+z-x=2a>0$ and $z+x-y=2b>0\Rightarrow x+y>z, y+z>x, z+x>y\Rightarrow x,y,z$ form

sides of a triangle with semiperimeter, circumradius and inradius $= s, R, r$ (say);

so $2\sum_{cyc} a = \sum_{cyc} x = 2s \Rightarrow \sum_{cyc} a = s \to (1) \Rightarrow a = s - x, b = s - y, c = s - z$ and such substitutions $\Rightarrow \sum_{cyc} ab = \sum_{cyc}(s-x)(s-y) \Rightarrow \sum_{cyc} ab = 4Rr + r^2 \to (2)$ and

$$\sum_{cyc} a^2 = \left(\sum_{cyc} a\right)^2 - 2\sum_{cyc} ab \overset{\text{via (1) and (2)}}{=} s^2 - 2(4Rr + r^2)$$

$$\Rightarrow \sum_{cyc} a^2 = s^2 - 8Rr - 2r^2 \to (3) \text{ and finally, } \sum_{cyc} a^3 =$$

$$\left(\sum_{cyc} a\right)^3 - 3(a+b)(b+c)(c+a) \overset{\text{via (1)}}{=} s^3 - 12Rrs \Rightarrow$$

$$\sum_{cyc} a^3 = s(s^2 - 12Rr) \to (4) \therefore \text{via } (1),(2),(3) \text{ and } (4), (**) \Leftrightarrow$$

$$8(s^2 - 8Rr - 2r^2)^2 \overset{?}{\geq} (4Rr + r^2)s^2 + 5s^2(s^2 - 12Rr)$$

$$\Leftrightarrow 3s^4 - (72Rr + 33r^2)s^2 + 32r^2(4R + r)^2 \underset{(***)}{\overset{?}{\geq}} 0 \text{ and } \because 3(s^2 - 16Rr + 5r^2)^2$$

$$\overset{\text{Gerretsen}}{\geq} 0 \therefore \text{in order to prove } (***), \text{it suffices to prove :}$$

$$\text{LHS of } (***) \overset{?}{\geq} 3(s^2 - 16Rr + 5r^2)^2$$

$$\Leftrightarrow (24R - 63r)s^2 \underset{(****)}{\overset{?}{\geq}} r(256R^2 - 736Rr + 43r^2)$$

Case 1 $24R - 63r \geq 0$ and then : LHS of $(****) \overset{\text{Gerretsen}}{\geq}$

$$(24R - 63r)(16Rr - 5r^2) \overset{?}{\geq} r(256R^2 - 736Rr + 43r^2)$$

$$\Leftrightarrow 8r^2(16R^2 - 49Rr + 34r^2) \overset{?}{\geq} 0 \Leftrightarrow 8r^2(R - 2r)(16R - 17r) \overset{?}{\geq} 0$$

$$\to \text{true} \because R \overset{\text{Euler}}{\geq} 2r \Rightarrow (****) \text{ is true}$$

Case 2 $24R - 63r < 0$ and then : LHS of $(****) \overset{\text{Gerretsen}}{\geq}$

$$(24R - 63r)(4R^2 + 4Rr + 3r^2) \overset{?}{\geq} r(256R^2 - 736Rr + 43r^2)$$

$$\Leftrightarrow 24t^3 - 103t^2 + 139t - 58 \overset{?}{\geq} 0 \left(t = \frac{R}{r}\right) \Leftrightarrow (t-2)\big((t-2)(24t-7) + 15\big) \overset{?}{\geq} 0$$

$$\to \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (****) \text{ is true and combining both cases, } (****) \Rightarrow$$

$$(***) \text{ is true } \forall \text{ triangle}_{s,R,r} \Rightarrow (**) \Rightarrow (*) \text{ is true } \therefore \sum_{cyc} \frac{a^3}{a^2 + \lambda b} \geq \frac{3}{\lambda + 1}$$

$$\forall a, b, c > 0 \,\middle|\, a + b + c = 3 \wedge \lambda \geq \frac{3}{5}, \text{"="} \text{ iff } a = b = c = 1 \text{ (QED)}$$