ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0, ab + bc + ca + 2abc = 1$, then :

$$\sum_{cyc} \frac{1}{a^5} \geq 2 \sum_{cyc} \frac{1}{b^2c^2}$$

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$$\text{We shall first prove the following : } \sum_{cyc} \frac{1}{a^{2n+1}} \geq 2 \sum_{cyc} \frac{1}{b^n c^n}$$

$$\forall\, a, b, c > 0 \,\big|\, ab + bc + ca + 2abc = 1 \text{ and } n \in \mathbb{N}$$

$$\text{We have : } \sum_{cyc} \big((1+b)(1+c)\big) = 2(1+a)(1+b)(1+c) \Rightarrow \sum_{cyc} \frac{1}{1+a} = 2$$

$$\Rightarrow \sum_{cyc} \frac{1}{1+\frac{1}{x}} = 2 \left(x = \frac{1}{a}, y = \frac{1}{b}, z = \frac{1}{c}\right) \Rightarrow \sum_{cyc} \frac{x+1-1}{x+1} = 2$$

$$\Rightarrow 3 - 2 = 1 = \sum_{cyc} \frac{1}{1+x} \overset{\text{Bergstrom}}{\geq} \frac{9}{\sum_{cyc} x + 3} \Rightarrow \sum_{cyc} x \geq 6 \to (\text{m}) \text{ and}$$

$$\sum_{cyc} \frac{1}{a^{2n+1}} \geq 2 \sum_{cyc} \frac{1}{b^n c^n} \text{ becomes : } \sum_{cyc} x^{2n+1} \geq 2 \sum_{cyc} y^n z^n \to (*)$$

$$\boxed{\text{Case 1}}\ n = 0 \text{ and then : } (*) \text{ is equivalent to : } \sum_{cyc} x \geq 6, \text{ which is true via (m)}$$

$$\Rightarrow (*) \text{ is true}$$

$$\boxed{\text{Case 2}}\ n \in \mathbb{N}^* \text{ and then : } \sum_{cyc} x^{2n+1} \overset{\text{Chebyshev}}{\geq} \frac{1}{3}\left(\sum_{cyc} x\right)\left(\sum_{cyc} x^{2n}\right) \overset{\text{via (m)}}{\geq}$$

$$2\left(\sum_{cyc} (x^n)^2\right) \geq 2 \sum_{cyc} y^n z^n \Rightarrow (*) \text{ is true and hence, combining both cases,}$$

$$(*) \text{ is true } \forall\, x, y, z > 0 \,\Big|\, \sum_{cyc} \frac{1}{1+x} = 1 \text{ and } n \in \mathbb{N}, '' ='' \text{ iff } x = y = z = 2$$

$$\therefore \sum_{cyc} \frac{1}{a^{2n+1}} \geq 2 \sum_{cyc} \frac{1}{b^n c^n}\ \forall\, a, b, c > 0 \,\big|\, ab + bc + ca + 2abc = 1 \text{ and } n \in \mathbb{N},$$

$$'' ='' \text{ iff } a = b = c = \frac{1}{2} \text{ and putting } n = 2, \text{ we get :}$$

$$\sum_{cyc} \frac{1}{a^5} \geq 2 \sum_{cyc} \frac{1}{b^2c^2}\ \forall\, a, b, c > 0 \,\big|\, ab + bc + ca + 2abc = 1,$$

$$'' ='' \text{ iff } a = b = c = \frac{1}{2}\ \text{(QED)}$$