If $a, b, c > 0$, $ab + bc + ca + 2abc = 1$ and $n \in \mathbb{N}$ then :

$$\sum_{cyc} \frac{1}{a^{2n+1}} \geq 2 \sum_{cyc} \frac{1}{b^n c^n}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

We have : $\sum_{cyc} \big((1+b)(1+c)\big) = 2(1+a)(1+b)(1+c) \Rightarrow \sum_{cyc} \frac{1}{1+a} = 2$

$\Rightarrow \sum_{cyc} \frac{1}{1 + \frac{1}{x}} = 2 \ \left(x = \frac{1}{a}, y = \frac{1}{b}, z = \frac{1}{c}\right) \Rightarrow \sum_{cyc} \frac{x + 1 - 1}{x + 1} = 2$

$\Rightarrow 3 - 2 = 1 = \sum_{cyc} \frac{1}{1+x} \overset{\text{Bergstrom}}{\geq} \frac{9}{\sum_{cyc} x + 3} \Rightarrow \sum_{cyc} x \geq 6 \to (m)$ and

$\sum_{cyc} \frac{1}{a^{2n+1}} \geq 2 \sum_{cyc} \frac{1}{b^n c^n}$ becomes : $\sum_{cyc} x^{2n+1} \geq 2 \sum_{cyc} y^n z^n \to (*)$

$\boxed{\text{Case 1}}$ $n = 0$ and then : $(*)$ is equivalent to : $\sum_{cyc} x \geq 6$, which is true via (m)

$\Rightarrow (*)$ is true

$\boxed{\text{Case 2}}$ $n \in \mathbb{N}^*$ and then : $\sum_{cyc} x^{2n+1} \overset{\text{Chebyshev}}{\geq} \frac{1}{3} \left(\sum_{cyc} x\right) \left(\sum_{cyc} x^{2n}\right) \overset{\text{via (m)}}{\geq}$

$2 \left(\sum_{cyc} (x^n)^2\right) \geq 2 \sum_{cyc} y^n z^n \Rightarrow (*)$ is true and hence, combining both cases,

$(*)$ is true $\forall \ x, y, z > 0 \ \big| \ \sum_{cyc} \frac{1}{1+x} = 1$ and $n \in \mathbb{N},'' ='' $ iff $x = y = z = 2$

$\therefore \sum_{cyc} \frac{1}{a^{2n+1}} \geq 2 \sum_{cyc} \frac{1}{b^n c^n} \ \forall \ a, b, c > 0 \ \big| \ ab + bc + ca + 2abc = 1$ and $n \in \mathbb{N}$,

$'' ='' $ iff $a = b = c = \frac{1}{2}$ (QED)