ROMANIAN MATHEMATICAL MAGAZINE

If $x, y, z > 0, x^3 + y^3 + z^3 + 3xyz = 6$ *and* $\lambda \geq 0$, then :

$$(xy + yz + zx)^3 + \lambda xyz \leq 27 + \lambda$$

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$$6 = \sum_{cyc} x^3 + 3xyz \overset{A-G}{\geq} 6xyz \Rightarrow 1 - xyz \geq 0 \text{ and } \because \lambda \geq 0 \therefore \lambda(1 - xyz) \geq 0$$

$$\Rightarrow \text{it suffices to prove : } 27 \geq \left(\sum_{cyc} xy\right)^3 \Leftrightarrow \frac{27}{36}\left(\sum_{cyc} x^3 + 3xyz\right)^2 \geq \left(\sum_{cyc} xy\right)^3$$

$$\left(\because 6 = \sum_{cyc} x^3 + 3xyz\right) \Leftrightarrow 3\left(\sum_{cyc} x^3 + 3xyz\right)^2 \overset{(*)}{\geq} 4\left(\sum_{cyc} xy\right)^3$$

Assigning $y + z = a, z + x = b, x + y = c \Rightarrow a + b - c = 2z > 0, b + c - a = 2x > 0$ *and* $c + a - b = 2y > 0 \Rightarrow a + b > c, b + c > a, c + a > b \Rightarrow a, b, c$ form sides of a triangle with semiperimeter, circumradius *and* inradius $= s, R, r$ (*say*)

$$\text{yielding } 2\sum_{cyc} x = \sum_{cyc} a = 2s \Rightarrow \sum_{cyc} x \overset{(\bullet)}{=} s \Rightarrow x = s - a, y = s - b, z = s - c$$

$$\therefore xyz \overset{(\bullet\bullet)}{=} r^2 s \text{ and}, \sum_{cyc} xy = \sum_{cyc} (s - a)(s - b) = 4Rr + r^2 \Rightarrow \sum_{cyc} xy \overset{(\bullet\bullet\bullet)}{=} 4Rr + r^2$$

$$\text{and also}, \sum_{cyc} x^3 = \left(\sum_{cyc} x\right)^3 - 3(x + y)(y + z)(z + x) \overset{\text{via } (\bullet)}{=} s^3 - 3.4Rrs$$

$$\Rightarrow \sum_{cyc} x^3 \overset{(\bullet\bullet\bullet\bullet)}{=} s^3 - 12Rrs$$

Via $(\bullet\bullet), (\bullet\bullet\bullet)$ *and* $(\bullet\bullet\bullet\bullet), (*)$ becomes : $3(s^3 - 12Rrs + 3r^2 s)^2 \geq 4(4Rr + r^2)^3$

$$\Leftrightarrow 3s^2(s^2 - 12Rr + 3r^2)^2 \overset{(**)}{\geq} 4r^3(4R + r)^3$$

Now, $s^2 \overset{\text{Gerretsen}}{\geq} 12Rr + 3r^2 + 4r(R - 2r) \overset{\text{Euler}}{\geq} 3r(4R + r)$ *and* so, in order to prove $(**)$, it suffices to prove : $9(s^2 - 12Rr + 3r^2)^2 \geq 4r^2(4R + r)^2$
$\Leftrightarrow 3(s^2 - 12Rr + 3r^2) \geq 2r(4R + r) \Leftrightarrow 3s^2 \geq 44Rr - 7r^2 \rightarrow$ true

$\because 3s^2 \overset{\text{Gerretsen}}{\geq} 44Rr - 7r^2 + 4r(R - 2r) \overset{\text{Euler}}{\geq} 44Rr - 7r^2 \Rightarrow (**) \Rightarrow (*)$ is true

$\therefore (xy + yz + zx)^3 + \lambda xyz \leq 27 + \lambda \ \forall\, x, y, z > 0 \,\middle|\, x^3 + y^3 + z^3 + 3xyz = 6$ *and* $\lambda \geq 0$, " = " iff $x = y = z = 1$ (QED)