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If $a,b,c>0, a+b+c=3$ and $\lambda \geq \frac{1}{2}$, then :

$$\lambda \sum_{cyc} a^3 \geq \sum_{cyc} a^2 + 3(\lambda - 1)$$

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$$\lambda \sum_{cyc} a^3 \geq \sum_{cyc} a^2 + 3(\lambda-1) \Leftrightarrow \lambda\left(\sum_{cyc} a^3 - \frac{3}{27}\left(\sum_{cyc} a\right)^3\right) \geq$$

$$\frac{1}{3}\left(\sum_{cyc} a\right)\left(\sum_{cyc} a^2\right) - \frac{3}{27}\left(\sum_{cyc} a\right)^3 \left(\because \sum_{cyc} a = 3\right)$$

$$\Leftrightarrow \lambda\left(9\sum_{cyc} a^3 - \left(\sum_{cyc} a\right)^3\right) \geq 3\left(\sum_{cyc} a\right)\left(\sum_{cyc} a^2\right) - \left(\sum_{cyc} a\right)^3 \text{ and } \because \lambda \geq \frac{1}{2}$$

$$\therefore \text{ it suffices to prove : } 9\sum_{cyc} a^3 - \left(\sum_{cyc} a\right)^3 \geq 6\left(\sum_{cyc} a\right)\left(\sum_{cyc} a^2\right) - 2\left(\sum_{cyc} a\right)^3$$

$$\Leftrightarrow 4\sum_{cyc} a^3 + 6abc \overset{(*)}{\geq} 3\sum_{cyc} a^2b + 3\sum_{cyc} ab^2$$

$$\text{Now, } 4\sum_{cyc} a^3 + 12abc \underset{①}{\overset{\text{Schur}}{\geq}} 4\sum_{cyc} a^2b + 4\sum_{cyc} ab^2 \text{ and}$$

$$\sum_{cyc} a^2b + \sum_{cyc} ab^2 \underset{②}{\overset{A-G}{\geq}} 6abc \text{ and via } ① + ②, (*) \text{ is true}$$

$$\therefore \lambda \sum_{cyc} a^3 \geq \sum_{cyc} a^2 + 3(\lambda-1)\ \forall\, a,b,c > 0 \,\middle|\, a+b+c=3 \text{ and } \lambda \geq \frac{1}{2},$$

$$'' ='' \text{ iff } a = b = c = 1 \text{ (QED)}$$