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If $a, b, c, \lambda > 0$ then :

$$(a-\lambda)^2+(b-\lambda)^2+(c-\lambda)^2 \geq \frac{ab+bc+ca}{\lambda+1}-3\lambda$$

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Let $x=\frac{a}{\lambda+1}, y=\frac{b}{\lambda+1}, z=\frac{c}{\lambda+1}$ and then :

$$\boxed{(a-\lambda)^2+(b-\lambda)^2+(c-\lambda)^2-\left(\frac{ab+bc+ca}{\lambda+1}-3\lambda\right)}$$

$$=(\lambda+1)^2\sum_{cyc}\left(x-\frac{\lambda}{\lambda+1}\right)^2-(\lambda+1)\sum_{cyc}xy+3\lambda$$

$$\geq \frac{(\lambda+1)^2}{3}.\left(\sum_{cyc}x-\frac{3\lambda}{\lambda+1}\right)^2-\frac{(\lambda+1)}{3}.\left(\sum_{cyc}x\right)^2+3\lambda$$

$$=\frac{1}{3}\left((\lambda+1)^2\left(t^2-\frac{6\lambda t}{\lambda+1}+\frac{9\lambda^2}{(\lambda+1)^2}\right)-(\lambda+1)t^2+9\lambda\right)\left(t=\sum_{cyc}x\right)$$

$$=\frac{1}{3}\left(\left((\lambda+1)^2-(\lambda+1)\right)t^2-6\lambda(\lambda+1)t+9\lambda^2+9\lambda\right)$$

$$=\frac{\lambda^2+\lambda}{3}.(t^2-6t+9)\boxed{=\frac{\lambda^2+\lambda}{3}.(t-3)^2\geq 0}(\because \lambda>0)$$

$$\therefore (a-\lambda)^2+(b-\lambda)^2+(c-\lambda)^2 \geq \frac{ab+bc+ca}{\lambda+1}-3\lambda \ \forall a,b,c,\lambda>0,$$

$$''='' \text{ iff } x=y=z=1 \Rightarrow \text{iff } \frac{a}{\lambda+1}=\frac{b}{\lambda+1}=\frac{c}{\lambda+1}=1$$

$$\Rightarrow \text{iff } a=b=c=\lambda+1 \text{ (QED)}$$