ROMANIAN MATHEMATICAL MAGAZINE

If $0 < x, y, z < 2$ with $xy + yz + zx = 3$ and $\lambda \geq 3$ then :

$$\sqrt{\frac{yz}{x^3 - x + \lambda}} + \sqrt{\frac{zx}{y^3 - y + \lambda}} + \sqrt{\frac{xy}{z^3 - z + \lambda}} \leq \frac{3}{\sqrt{\lambda}}$$

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We will first prove the lemma that for all $x \in (0, 2)$ and $\lambda \geq 3$, we have

$$\frac{1}{x^3 - x + \lambda} \leq \frac{\lambda + 2 - 2x}{\lambda^2}.$$

The inequality is equivalent to

$$(\lambda + 2 - 2x)(x^3 - x + \lambda) - \lambda^2 \geq 0 \text{ or } -2x^4 + (\lambda + 2)x^3 + 2x^2 - (3\lambda + 2)x + 2\lambda \geq 0$$

$$\text{or } (x - 1)^2[(\lambda - 3)(2 + x) + (2 - x)(3 + 2x)] \geq 0,$$

which is true and the proof of the lemma is complete.

Now, by using the CBS inequality and the lemma above, we have

$$\sum_{cyc} \sqrt{\frac{yz}{x^3 - x + \lambda}} \leq \sqrt{\sum_{cyc} yz . \sum_{cyc} \frac{1}{x^3 - x + \lambda}} \leq \sqrt{3 \sum_{cyc} \frac{\lambda + 2 - 2x}{\lambda^2}}$$

$$= \sqrt{3 . \frac{3(\lambda + 2) - 2(x + y + z)}{\lambda^2}}$$

$$\leq \sqrt{3 . \frac{3\lambda + 6 - 2\sqrt{3(xy + yz + zx)}}{\lambda^2}} = \frac{3}{\sqrt{\lambda}},$$

as desired. Equality holds iff $x = y = z = 1$.