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If $a, b, c > 0, abc = 1$ and $\lambda \geq 1$, then:

$$\sum_{cyc} \frac{1}{a^3 + b^3 + \lambda} \leq \frac{3}{\lambda + 2}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} \frac{1}{a^3 + b^3 + \lambda} \leq \sum_{cyc} \frac{1}{ab(a+b) + \lambda} \overset{abc=1}{=} \sum_{cyc} \frac{1}{\frac{a+b}{c} + \lambda} = \sum_{cyc} \frac{1}{x + \lambda}$$

$$\left(x = \frac{a+b}{c}, y = \frac{b+c}{a}, z = \frac{c+a}{b}\right) = \frac{1}{\lambda} \sum_{cyc} \frac{\lambda + x - x}{x + \lambda} = \frac{3}{\lambda} - \frac{1}{\lambda} \sum_{cyc} \frac{x}{x + \lambda} \overset{?}{\leq} \frac{3}{\lambda + 2}$$

$$\Leftrightarrow \sum_{cyc} \frac{x}{x + \lambda} \overset{?}{\geq} \frac{6}{\lambda + 2} \Leftrightarrow \frac{\sum_{cyc}(x(y+\lambda)(z+\lambda))}{(x+\lambda)(y+\lambda)(z+\lambda)} \overset{?}{\geq} \frac{6}{\lambda + 2}$$

$$\Leftrightarrow \frac{3xyz + 2\lambda \sum_{cyc} xy + \lambda^2 \sum_{cyc} x}{xyz + \lambda^3 + \lambda \sum_{cyc} xy + \lambda^2 \sum_{cyc} x} \overset{?}{\geq} \frac{6}{\lambda + 2} \Leftrightarrow$$

$$3\lambda xyz + 2\lambda^2 \sum_{cyc} xy + \lambda^3 \sum_{cyc} x + 6xyz + 4\lambda \sum_{cyc} xy + 2\lambda^2 \sum_{cyc} x$$
$$\overset{?}{\geq} 6xyz + 6\lambda \sum_{cyc} xy + 6\lambda^2 \sum_{cyc} x + 6\lambda^3$$

$$\Leftrightarrow \boxed{\lambda^3 \left(\sum_{cyc} x - 6\right) + \lambda^2 \left(2 \sum_{cyc} xy - 4 \sum_{cyc} x\right) + \lambda \left(3xyz - 2 \sum_{cyc} xy\right) \underset{(*)}{\overset{?}{\geq}} 0}$$

$$\bullet \sum_{cyc} x = \sum_{cyc} \frac{a+b}{c} \overset{abc=1}{=} \sum_{cyc} \left(ab \left(\sum_{cyc} a - c\right)\right) = \left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right) - 3abc$$
$$\rightarrow (1)$$

$$\bullet \sum_{cyc} xy = \sum_{cyc} \frac{(a+b)(b+c)}{ca} \overset{abc=1}{=} \sum_{cyc} \left(b\left(b^2 + \sum_{cyc} ab\right)\right) = \sum_{cyc} a^3 +$$
$$\left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right) \rightarrow (2)$$

$$\bullet\ xyz = \prod_{cyc} \frac{a+b}{c} \overset{abc=1}{=} \prod_{cyc} (a+b) \rightarrow (3) \therefore \text{via } (1), (2) \text{ and } (3), (*) \Leftrightarrow$$

$$\lambda^3\left(\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)-9abc\right)$$
$$+2\lambda^2\left(\sum_{cyc}a^3+\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)-2\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)+6abc\right)$$
$$+\lambda\left(3\prod_{cyc}(a+b)-2\sum_{cyc}a^3-2\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)\right)\geq 0$$
$$\Leftrightarrow \lambda^3\left(\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)-9abc\right)$$
$$+2\lambda^2\left(\sum_{cyc}a^3+6abc-\left(3abc+\sum_{cyc}a^2b+\sum_{cyc}ab^2\right)\right)$$
$$+\lambda\left(3\prod_{cyc}(a+b)-2\sum_{cyc}a^3-2\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)\right)\geq 0$$
$$\Leftrightarrow \boxed{\begin{array}{c}\lambda^3\left(\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)-9abc\right)+2\lambda^2\left(\sum_{cyc}a^3+3abc-\left(\sum_{cyc}a^2b+\sum_{cyc}ab^2\right)\right)\\+\lambda\left(3\prod_{cyc}(a+b)-2\sum_{cyc}a^3-2\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)\right)\overset{(**)}{\geq}0\end{array}}$$

Now, $\lambda\geq 1\Rightarrow\lambda^3,\lambda^2\geq\lambda$ and $\because\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)-9abc\overset{A-G}{\geq}0$ and

$\sum_{cyc}a^3+3abc-\left(\sum_{cyc}a^2b+\sum_{cyc}ab^2\right)\overset{Schur}{\geq}0\ \therefore$ LHS of $(**)\geq$

$$\lambda\left(\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)-9abc\right)+2\lambda\left(\sum_{cyc}a^3+3abc-\left(\sum_{cyc}a^2b+\sum_{cyc}ab^2\right)\right)$$
$$+\lambda\left(3\prod_{cyc}(a+b)-2\sum_{cyc}a^3-2\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)\right)$$
$$=\lambda\left(\begin{array}{c}3\prod_{cyc}(a+b)-\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)+2\left(3abc+3abc-\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)\right)\\-9abc\end{array}\right)$$

$$= \lambda\left(\begin{array}{c}3\left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right) - 3abc - \left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right) + 12abc \\ -2\left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right) - 9abc\end{array}\right) = 0$$

$$\Rightarrow (**) \Rightarrow (*) \text{ is true } \therefore \sum_{cyc} \frac{1}{a^3 + b^3 + \lambda} \le \frac{3}{\lambda + 2} \; \forall\, a, b, c > 0 \,\middle|\, abc = 1$$

$$\text{and } \forall\, \lambda \ge 1, '' ='' \text{ iff } (a = b = c = \lambda = 1) \text{ (QED)}$$