ROMANIAN MATHEMATICAL MAGAZINE

If $x, y, z > 0, xyz = 1$ and $1 \le \lambda \le 2$, then :

$$\sum_{cyc} \frac{x}{x^2+\lambda} \le \frac{3}{\lambda+1}$$

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Solution 1 by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} \frac{x}{x^2+\lambda} \overset{xyz=1}{=} \sum_{cyc} \frac{1}{yz(x^2+\lambda)} \overset{xyz=1}{=} \sum_{cyc} \frac{1}{x+\lambda yz} = \sum_{cyc} \frac{1}{x+yz+(\lambda-1)yz}$$

$$\overset{A-G}{\le} \sum_{cyc} \frac{1}{2.\sqrt{xyz}+(\lambda-1)yz} \overset{xyz=1}{=} \sum_{cyc} \frac{1}{2+ta} \begin{pmatrix} 0 \le t = \lambda - 1 \le 1 \text{ and} \\ yz = a, zx = b, xy = c \end{pmatrix}$$

$$\overset{?}{\le} \frac{3}{\lambda+1} = \frac{3}{2+t}$$

$$\Leftrightarrow 3(2+ta)(2+tb)(2+tc) \overset{?}{\ge} (t+2)\begin{pmatrix} (2+ta)(2+tb)+ \\ (2+tb)(2+tc)+(2+tc)(2+ta) \end{pmatrix}$$

$$\overset{abc=1}{\Leftrightarrow} \boxed{t^3\left(3-\sum_{cyc} ab\right) + t^2\left(4\sum_{cyc} ab - 4\sum_{cyc} a\right) + t\left(4\sum_{cyc} a - 12\right) \underset{(*)}{\overset{?}{\ge}} 0}$$

$$\text{Now, } t^3\left(3-\sum_{cyc} ab\right) \overset{?}{\ge} t\left(3-\sum_{cyc} ab\right) \Leftrightarrow (t^3-t)\left(3-\sum_{cyc} ab\right) \overset{?}{\ge} 0 \to \text{true}$$

$$\because 0 \le t \le 1 \Rightarrow t^3 - t \le 0 \text{ and } 3 - \sum_{cyc} ab \overset{A-G}{\le} 3 - 3.\sqrt[3]{a^2b^2c^2} \overset{abc=1}{=} 0$$

$$\therefore t^3\left(3-\sum_{cyc} ab\right) \overset{(\bullet)}{\ge} t\left(3-\sum_{cyc} ab\right)$$

$$\text{Again, } t^3\left(3-\sum_{cyc} ab\right) \overset{?}{\ge} t^2\left(3-\sum_{cyc} ab\right) \Leftrightarrow (t^3-t^2)\left(3-\sum_{cyc} ab\right) \overset{?}{\ge} 0 \to \text{true}$$

$$\because 0 \le t \le 1 \Rightarrow t^3 - t^2 \le 0 \text{ and } 3 - \sum_{cyc} ab \overset{A-G}{\le} 3 - 3.\sqrt[3]{a^2b^2c^2} \overset{abc=1}{=} 0$$

$$\therefore t^3\left(3-\sum_{cyc} ab\right) \overset{(\bullet\bullet)}{\ge} t^2\left(3-\sum_{cyc} ab\right)$$

$$\text{Also, } t^2\left(3-\sum_{cyc} a\right) \overset{?}{\ge} t\left(3-\sum_{cyc} a\right) \Leftrightarrow (t^2-t)\left(3-\sum_{cyc} a\right) \overset{?}{\ge} 0 \to \text{true}$$

$$\because 0 \le t \le 1 \Rightarrow t^2 - t \le 0 \text{ and } 3 - \sum_{cyc} a \overset{A-G}{\le} 3 - 3.\sqrt[3]{abc} \overset{abc=1}{=} 0$$

$$\therefore t^2\left(3-\sum_{cyc} a\right) \overset{(\bullet\bullet\bullet)}{\geq} t\left(3-\sum_{cyc} a\right)$$

$\boxed{Case\ 1}$ $\boxed{\sum_{cyc} a \geq \sum_{cyc} ab}$ and then : $t^2\left(4\sum_{cyc} ab - 4\sum_{cyc} a\right) \overset{?}{\geq}$

$$t\left(4\sum_{cyc} ab - 4\sum_{cyc} a\right) \Leftrightarrow (t^2-t)\left(4\sum_{cyc} ab - 4\sum_{cyc} a\right) \overset{?}{\geq} 0 \rightarrow \text{true}$$

$$\because 0 \leq t \leq 1 \Rightarrow t^2 - t \leq 0 \text{ and } 4\sum_{cyc} ab - 4\sum_{cyc} a \leq 0$$

$$\therefore t^2\left(4\sum_{cyc} ab - 4\sum_{cyc} a\right) \overset{(\bullet\bullet\bullet\bullet)}{\geq} t\left(4\sum_{cyc} ab - 4\sum_{cyc} a\right) \therefore (\bullet) + (\bullet\bullet\bullet\bullet) \Rightarrow \text{LHS of } (*)$$

$$\geq t\left(3-\sum_{cyc} ab\right) + t\left(4\sum_{cyc} ab - 4\sum_{cyc} a\right) + t\left(4\sum_{cyc} a - 12\right)$$

$$= t\left(3-\sum_{cyc} ab + 4\sum_{cyc} ab - 4\sum_{cyc} a + 4\sum_{cyc} a - 12\right) = 3t\left(\sum_{cyc} ab - 3\right) \geq 0$$

$$\because t \geq 0 \text{ and } \sum_{cyc} ab \overset{A-G}{\geq} 3.\sqrt[3]{a^2b^2c^2} \overset{abc=1}{=} 3 \Rightarrow \sum_{cyc} ab - 3 \geq 0 \Rightarrow \boxed{(*) \text{ is true}}$$

$\boxed{Case\ 2}$ $\boxed{\sum_{cyc} ab \geq \sum_{cyc} a}$ and then : LHS of $(*) \overset{\text{via } (\bullet\bullet)}{\geq} t^2\left(3-\sum_{cyc} ab\right)$

$$+t^2\left(4\sum_{cyc} ab - 4\sum_{cyc} a\right) + t\left(4\sum_{cyc} a - 12\right)$$

$$= t^2\left(3\sum_{cyc} ab - 3\sum_{cyc} a\right) + t^2\left(3-\sum_{cyc} a\right) + t\left(4\sum_{cyc} a - 12\right)$$

$$\overset{\text{via } (\bullet\bullet\bullet)}{\geq} t^2\left(3\sum_{cyc} ab - 3\sum_{cyc} a\right) + t\left(3-\sum_{cyc} a\right) + t\left(4\sum_{cyc} a - 12\right)$$

$$= 3t^2\left(\sum_{cyc} ab - \sum_{cyc} a\right) + 3t\left(\sum_{cyc} a - 3\right) \rightarrow \text{true}$$

$$\because \sum_{cyc} ab \geq \sum_{cyc} a \text{ and } \sum_{cyc} a \overset{A-G}{\geq} 3.\sqrt[3]{abc} \overset{abc=1}{=} 3 \Rightarrow \sum_{cyc} a - 3 \geq 0 \Rightarrow \boxed{(*) \text{ is true}}$$

$\therefore$ combining both cases, $(*)$ is true $\forall\, a,b,c > 0 \mid abc = 1$ and $\forall\, t \in [0,1]$

$$\therefore \sum_{cyc} \frac{x}{x^2+\lambda} \leq \frac{3}{\lambda+1}\ \forall\, x,y,z > 0 \mid xyz = 1 \text{ and } \forall\, \lambda \in [1,2],$$

$$'' ='' \text{ iff } x = y = z = 1 \text{ (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\sum_{cyc}\frac{x}{x^2+\lambda}=\sum_{cyc}\frac{x}{x^2+1+(\lambda-1)}\overset{AM-GM}{\leq}\sum_{cyc}\frac{x}{2x+\lambda-1}=\sum_{cyc}\left(\frac{1}{2}-\frac{\lambda-1}{2(2x+\lambda-1)}\right)$$

$$=\frac{3}{2}-\frac{\lambda-1}{2}\sum_{cyc}\frac{yz}{2+(\lambda-1)yz}\overset{CBS}{\leq}\frac{3}{2}-\frac{(\lambda-1)\left(\sqrt{yz}+\sqrt{zx}+\sqrt{xy}\right)^2}{2[6+(\lambda-1)(yz+zx+xy)]}$$

$$=\frac{3}{2}-\frac{(\lambda-1)\left[xy+yz+zx+2\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)\right]}{2[6+(\lambda-1)(yz+zx+xy)]}\overset{AM-GM}{\leq}\frac{3}{2}-\frac{(\lambda-1)[xy+yz+zx+2.3]}{2[6+(\lambda-1)(yz+zx+xy)]}$$

$$=1+\frac{3(2-\lambda)}{6+(\lambda-1)(yz+zx+xy)}\overset{AM-GM}{\leq}1+\frac{3(2-\lambda)}{6+(\lambda-1).3}=\frac{3}{\lambda+1},$$

as desired. Equality holds iff $x=y=z=1$.