ROMANIAN MATHEMATICAL MAGAZINE

If $x, y, z > 0$, then :

$$\frac{xy + yz + zx}{x^2 + y^2 + z^2} + \frac{(x + y + z)^3}{9xyz} \geq 4$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

Assigning $y + z = a, z + x = b, x + y = c \Rightarrow a + b - c = 2z > 0, b + c - a = 2x > 0$ and $c + a - b = 2y > 0 \Rightarrow a + b > c, b + c > a, c + a > b$
$\Rightarrow a, b, c$ form sides of a triangle with semiperimeter, circumradius and inradius $= s, R, r$ (say) yielding $2\sum_{cyc} x = \sum_{cyc} a = 2s \Rightarrow \sum_{cyc} x \overset{(i)}{=} s$

$$\Rightarrow x = s - a, y = s - b, z = s - c$$

Via such substitutions, $xyz = (s-a)(s-b)(s-c) \Rightarrow xyz \overset{(ii)}{=} r^2 s$ and
$\sum_{cyc} xy = \sum_{cyc}(s-a)(s-b) = 4Rr + r^2 \Rightarrow \sum_{cyc} xy \overset{(1)}{=} 4Rr + r^2$

$$\Rightarrow \sum_{cyc} x^2 = \left(\sum_{cyc} x\right)^2 - 2\sum_{cyc} xy \overset{\text{via (i),(1)}}{=} s^2 - 2(4Rr + r^2)$$

$$\Rightarrow \sum_{cyc} x^2 \overset{(2)}{=} s^2 - 8Rr - 2r^2 \therefore \frac{xy + yz + zx}{x^2 + y^2 + z^2} + \frac{(x+y+z)^3}{9xyz} \overset{\text{via (i),(ii),(1),(2)}}{=}$$

$$\frac{4Rr + r^2}{s^2 - 8Rr - 2r^2} + \frac{s^3}{9r^2 s} = \frac{9r^2(4Rr + r^2) + s^2(s^2 - 8Rr - 2r^2)}{9r^2(s^2 - 8Rr - 2r^2)} \overset{?}{\geq} 4$$

$$\Leftrightarrow s^4 - (8Rr + 38r^2)s^2 + 81r^3(4R + r) \underset{(*)}{\overset{?}{\geq}} 0$$

Now, LHS of $(*) \overset{\text{Gerretsen}}{\geq} (8Rr - 43r^2)s^2 + 81r^3(4R + r) \overset{?}{\geq} 0$

$$\Leftrightarrow (8R - 43r)s^2 + 81r^2(4R + r) \underset{(**)}{\overset{?}{\geq}} 0$$

Case 1 $8R - 43r \geq 0$ and then : LHS of $(*) \geq 81r^2(4R + r) > 0$
$\Rightarrow (**)$ is true (strict inequality)

Case 2 $8R - 43r < 0$ and then : LHS of $(*) = -(43r - 8R)s^2 + 81r^2(4R + r)$
$\overset{\text{Gerretsen}}{\geq} -(43r - 8R)(4R^2 + 4Rr + 3r^2) + 81r^2(4R + r) \overset{?}{\geq} 0$

$$\Leftrightarrow 8t^3 - 35t^2 + 44t - 12 \overset{?}{\geq} 0 \left(t = \frac{R}{r}\right) \Leftrightarrow (8t - 3)(t - 2)^2 \overset{?}{\geq} 0 \rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2$$

$\Rightarrow (**)$ is true $\therefore$ combining cases $1, 2, (**) \Rightarrow (*)$ is true $\forall$ triangles of sides a, b, c

$$\Rightarrow \frac{xy + yz + zx}{x^2 + y^2 + z^2} + \frac{(x+y+z)^3}{9xyz} \geq 4 \ \forall\, x, y, z > 0, \text{''} = \text{''} \text{ iff } x = y = z \text{ (QED)}$$