ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ and $\lambda \geq \frac{1}{2}$, then :

$$\frac{a}{\sqrt{ab+\lambda b^2}}+\frac{b}{\sqrt{bc+\lambda c^2}}+\frac{c}{\sqrt{ca+\lambda a^2}} \geq \frac{3}{\sqrt{\lambda+1}}$$

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Assigning $b+c=x, c+a=y, a+b=z \Rightarrow x+y-z=2c>0, y+z-x=2a>0$ and $z+x-y=2b>0 \Rightarrow x+y>z, y+z>x, z+x>y \Rightarrow x, y, z$ form sides of a triangle with semiperimeter, circumradius and inradius $= s, R, r$ (say) yielding $2\sum_{cyc} a=\sum_{cyc} x=2s \Rightarrow \sum_{cyc} a=s \rightarrow (1) \Rightarrow a=s-x, b=s-y,$

$c=s-z$ and such substitutions $\Rightarrow \sum_{cyc} ab=\sum_{cyc}(s-x)(s-y)$

$$\Rightarrow \sum_{cyc} ab=4Rr+r^2 \rightarrow (2), \sum_{cyc} a^2=\left(\sum_{cyc} a\right)^2-2\sum_{cyc} ab \overset{\text{via (1) and (2)}}{=}$$

$$s^2-2(4Rr+r^2) \Rightarrow \sum_{cyc} a^2=s^2-8Rr-2r^2 \rightarrow (3)$$

$$\text{Now, } \frac{a}{\sqrt{ab+\lambda b^2}}+\frac{b}{\sqrt{bc+\lambda c^2}}+\frac{c}{\sqrt{ca+\lambda a^2}}=\sum_{cyc}\frac{a^2}{\sqrt{ab}.\sqrt{a^2+\lambda ab}} \overset{\text{Bergstrom}}{\geq}$$

$$\frac{\left(\sum_{cyc} a\right)^2}{\sum_{cyc}\left(\sqrt{ab}.\sqrt{a^2+\lambda ab}\right)} \overset{\text{CBS}}{\geq} \frac{\left(\sum_{cyc} a\right)^2}{\sqrt{\sum_{cyc} ab}.\sqrt{\sum_{cyc} a^2+\lambda\sum_{cyc} ab}} \overset{?}{\geq} \frac{3}{\sqrt{\lambda+1}}$$

$$\overset{\text{via (1),(2) and (3)}}{\Leftrightarrow} \frac{s^4}{(4Rr+r^2)(s^2-8Rr-2r^2+\lambda(4Rr+r^2))} \overset{?}{\geq} \frac{9}{\lambda+1}$$

$$\Leftrightarrow \lambda\left(s^4-9(4Rr+r^2)^2\right)+s^4-(36Rr+9r^2)(s^2-8Rr-2r^2) \underset{(*)}{\overset{?}{\geq}} 0$$

We have : $s^4-9(4Rr+r^2)^2=(s^2-12Rr-3r^2)(s^2+12Rr+3r^2) \overset{\text{Gerretsen + Euler}}{\geq} 0$ and $\because \lambda \geq \frac{1}{2} \therefore$ LHS of $(*) \geq$

$$\frac{1}{2}\left(s^4-9(4Rr+r^2)^2\right)+s^4-(36Rr+9r^2)(s^2-8Rr-2r^2) \overset{?}{\geq} 0$$

$$\Leftrightarrow s^4-(24Rr+6r^2)+9r^2(4R+r)^2 \overset{?}{\geq} 0 \Leftrightarrow (s^2-12Rr-3r^2)^2 \overset{?}{\geq} 0 \rightarrow \text{true}$$

$$\Rightarrow (*) \text{ is true} \therefore \frac{a}{\sqrt{ab+\lambda b^2}}+\frac{b}{\sqrt{bc+\lambda c^2}}+\frac{c}{\sqrt{ca+\lambda a^2}} \geq \frac{3}{\sqrt{\lambda+1}}$$

$\forall a, b, c > 0$ and $\lambda \geq \frac{1}{2}$, " = " iff $a=b=c$ (QED)