ROMANIAN MATHEMATICAL MAGAZINE

If $x, y, z > 0, x + y + z = 1$ and $\lambda \geq \frac{2}{27}$, then :

$$\sum_{cyc} \frac{\lambda - x^3}{x} \geq \frac{27\lambda - 1}{3}$$

Proposed by Marin Chirciu-Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

Assigning $y + z = a, z + x = b, x + y = c \Rightarrow a + b - c = 2z > 0, b + c - a = 2x > 0$ and $c + a - b = 2y > 0 \Rightarrow a + b > c, b + c > a, c + a > b \Rightarrow a, b, c$ form sides of a triangle with semiperimeter, circumradius and inradius $= s, R, r$ (say)

yielding $2\sum_{cyc} x = \sum_{cyc} a = 2s \Rightarrow \sum_{cyc} x \overset{(\bullet)}{=} s \Rightarrow x = s - a, y = s - b, z = s - c$

$\therefore xyz \overset{(\bullet\bullet)}{=} r^2 s$ and, $\sum_{cyc} xy = \sum_{cyc} (s-a)(s-b) = 4Rr + r^2 \Rightarrow \sum_{cyc} xy \overset{(\bullet\bullet\bullet)}{=} 4Rr + r^2$

and also, $\sum_{cyc} x^2 = \left(\sum_{cyc} x\right)^2 - 2\sum_{cyc} xy \overset{\text{via } (\bullet) \text{ and } (\bullet\bullet\bullet)}{=} s^2 - 2(4Rr + r^2)$

$$\Rightarrow \sum_{cyc} x^2 \overset{(\bullet\bullet\bullet\bullet)}{=} s^2 - 8Rr - 2r^2$$

$$\sum_{cyc} \frac{\lambda - x^3}{x} \geq \frac{27\lambda - 1}{3} \Leftrightarrow \lambda\left(\frac{\sum_{cyc} xy - 9xyz}{xyz}\right) \geq \sum_{cyc} x^2 - \frac{1}{3} \overset{x+y+z=1}{\Leftrightarrow}$$

$$\lambda\left(\frac{(\sum_{cyc} x)(\sum_{cyc} xy) - 9xyz}{xyz}\right) \boxed{\overset{(*)}{\geq}} \frac{\sum_{cyc} x^2}{(\sum_{cyc} x)^2} - \frac{1}{3}$$

Now, $\left(\sum_{cyc} x\right)\left(\sum_{cyc} xy\right) \overset{A-G}{\geq} 9xyz \Rightarrow \frac{(\sum_{cyc} x)(\sum_{cyc} xy) - 9xyz}{xyz} \geq 0$ and $\because \lambda \geq \frac{2}{27}$

$$\therefore \text{LHS of } (*) \geq \frac{2}{27}\left(\frac{(\sum_{cyc} x)(\sum_{cyc} xy) - 9xyz}{xyz}\right) \overset{?}{\geq} \frac{\sum_{cyc} x^2}{(\sum_{cyc} x)^2} - \frac{1}{3}$$

$$\overset{\text{via } (\bullet),(\bullet\bullet),(\bullet\bullet\bullet) \text{ and } (\bullet\bullet\bullet\bullet)}{\Leftrightarrow} \frac{2}{27} \cdot \frac{s(4Rr + r^2) - 9r^2 s}{r^2 s} \overset{?}{\geq} \frac{3(s^2 - 8Rr - 2r^2) - s^2}{3s^2}$$

$$\Leftrightarrow \frac{s^2(4R - 8r)}{9r} \boxed{\underset{(**)}{\overset{?}{\geq}}} s^2 - 12Rr - 3r^2$$

Again, $s^2 - 12Rr - 3r^2 \underbrace{\overset{\text{Gerretsen}}{\leq}}_{(i)} 4R^2 - 8Rr$ and

$$\frac{s^2(4R - 8r)}{9r} \underbrace{\overset{\text{Gerretsen}}{\geq}}_{(ii)} \frac{(4R - 8r)(16R - 5r)}{9}$$

$$\therefore \text{(i)},\text{(ii)} \Rightarrow \text{in order to prove } (**), \text{it suffices to prove}:$$

$$\frac{(4R-8r)(16R-5r)}{9} \ge 4R^2 - 8Rr \Leftrightarrow 4(R-2r)(7R-5r) \ge 0 \to \text{true via Euler}$$

$$\Rightarrow (**) \Rightarrow (*) \text{ is true} \therefore \sum_{cyc} \frac{\lambda - x^3}{x} \ge \frac{27\lambda - 1}{3}\ \forall x,y,z > 0 \,\middle|\, x+y+z=1$$

$$\text{and } \lambda \ge \frac{2}{27}, '' ='' \text{ iff } x = y = z = \frac{1}{3}\ \text{(QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

The given inequality can be rewritten as follows

$$\lambda\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}-9\right) \ge x^2+y^2+z^2-\frac{1}{3}.$$

Since $\frac{1}{x}+\frac{1}{y}+\frac{1}{z} \ge \frac{9}{x+y+z} = 9$, **so it suffices to prove that**

$$\frac{2}{27}\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}-9\right) \ge x^2+y^2+z^2-\frac{1}{3} \quad \text{or}$$

$$\frac{2}{27}\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) + 2(xy+yz+zx) \ge \frac{4}{3} \quad (1)$$

$$\blacksquare LHS_{(1)} \overset{AM-GM}{\ge} 2\sqrt{\frac{4}{27}\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)(xy+yz+zx)} = \frac{4}{3}\sqrt{\frac{(xy+yz+zx)^2}{3xyz}}$$

$$\ge \frac{4}{3}\sqrt{x+y+z} = \frac{4}{3}.$$

So the proof is complete. Equality holds iff $x = y = z = \frac{1}{3}$.