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If $a, \mathrm{b}, \mathrm{c} > 0$ such that : $a + \mathrm{b} + \mathrm{c} = 3$ and $\lambda \ge 0$, then :

$$\frac{a}{\mathrm{b}}(\mathrm{b}+\lambda\mathrm{c})+\frac{\mathrm{b}}{\mathrm{c}}(\mathrm{c}+\lambda a)+\frac{\mathrm{c}}{a}(a+\lambda\mathrm{b}) \ge \frac{3}{2}(1+(2\lambda+1)a\mathrm{bc})$$

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$$\frac{a}{\mathrm{b}}(\mathrm{b}+\lambda\mathrm{c})+\frac{\mathrm{b}}{\mathrm{c}}(\mathrm{c}+\lambda a)+\frac{\mathrm{c}}{a}(a+\lambda\mathrm{b}) \ge \frac{3}{2}(1+(2\lambda+1)a\mathrm{bc})$$

$$\Leftrightarrow \sum_{\mathrm{cyc}} a + \lambda \sum_{\mathrm{cyc}} \frac{\mathrm{c}a}{\mathrm{b}} \ge \frac{3}{2}+\frac{3}{2}(2\lambda+1)a\mathrm{bc}$$

$$\overset{a+\mathrm{b}+\mathrm{c}=3}{\Leftrightarrow} \frac{\left(3-\frac{3}{2}\right)}{27}\left(\sum_{\mathrm{cyc}} a\right)^3 + \frac{\lambda}{9a\mathrm{bc}}\left(\sum_{\mathrm{cyc}} a^2\mathrm{b}^2\right)\left(\sum_{\mathrm{cyc}} a\right)^2 \ge 3\lambda a\mathrm{bc} + \frac{3}{2}a\mathrm{bc}$$

$$\Leftrightarrow \left(\sum_{\mathrm{cyc}} a\right)^3 + \frac{2\lambda}{a\mathrm{bc}}\left(\sum_{\mathrm{cyc}} a^2\mathrm{b}^2\right)\left(\sum_{\mathrm{cyc}} a\right)^2 \ge 54\lambda a\mathrm{bc} + 27a\mathrm{bc} \to \text{true}$$

$$\because \left(\sum_{\mathrm{cyc}} a\right)^3 \overset{\mathrm{A-G}}{\ge} 27a\mathrm{bc} \text{ and } \because 2\lambda\left(\sum_{\mathrm{cyc}} a^2\mathrm{b}^2\right)\left(\sum_{\mathrm{cyc}} a\right)^2 \overset{\substack{\mathrm{A-G} \\ and \\ \because \lambda \ge 0}}{\ge}$$

$$2\lambda\left(3\sqrt[3]{a^4\mathrm{b}^4\mathrm{c}^4}\right)\left(3\sqrt[3]{a\mathrm{bc}}\right)^2 = 54\lambda a^2\mathrm{b}^2\mathrm{c}^2 \Rightarrow \frac{2\lambda}{a\mathrm{bc}}\left(\sum_{\mathrm{cyc}} a^2\mathrm{b}^2\right)\left(\sum_{\mathrm{cyc}} a\right)^2 \ge 54\lambda a\mathrm{bc}$$

$$\therefore \frac{a}{\mathrm{b}}(\mathrm{b}+\lambda\mathrm{c})+\frac{\mathrm{b}}{\mathrm{c}}(\mathrm{c}+\lambda a)+\frac{\mathrm{c}}{a}(a+\lambda\mathrm{b}) \ge \frac{3}{2}(1+(2\lambda+1)a\mathrm{bc})$$

$$\forall\, a,\mathrm{b},\mathrm{c} > 0 \,\big|\, a+\mathrm{b}+\mathrm{c} = 3, '' ='' \text{ iff } a = \mathrm{b} = \mathrm{c} = 1 \text{ (QED)}$$