ROMANIAN MATHEMATICAL MAGAZINE

If $x,y,z,\lambda>0, x+y+z=3$ then:

$$1)\ \frac{1}{(x+\lambda)^3}+\frac{1}{(y+\lambda)^3}+\frac{1}{(z+\lambda)^3}\ge\frac{3}{(\lambda+1)^3}$$

$$2)\ \frac{x}{(y+\lambda)^3}+\frac{y}{(z+\lambda)^3}+\frac{z}{(x+\lambda)^3}\ge\frac{3}{(\lambda+1)^3}$$

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Solution by Tapas Das-India

$$1)\ \frac{1}{(x+\lambda)^3}+\frac{1}{(y+\lambda)^3}+\frac{1}{(z+\lambda)^3}\ge\frac{3}{(\lambda+1)^3}$$

$$Let\ f(p)=\frac{1}{(p+\lambda)^3},p\in(0,3) and\ \lambda\ge 0, f'(p)=-\frac{3}{(p+\lambda)^4},$$

$$f''(p)=\frac{12}{(p+\lambda)^5}>0\ so\ f\ is\ convex\in(0,3). Using\ Jensen\ inequality$$

$$f(x)+f(y)+f(z)\ge 3f\left(\frac{x+y+z}{3}\right)=3f(1)\ or$$

$$\frac{1}{(x+\lambda)^3}+\frac{1}{(y+\lambda)^3}+\frac{1}{(z+\lambda)^3}\ge\frac{3}{(1+\lambda)^3}, equality\ holds\ a=b=c$$

$$2)\ \frac{x}{(y+\lambda)^3}+\frac{y}{(z+\lambda)^3}+\frac{z}{(x+\lambda)^3}\ge\frac{3}{(\lambda+1)^3}$$

$$LHS=\sum\frac{x^4}{(xy+\lambda x)^3}\overset{Radon}{\ge}\frac{(x+y+z)^4}{\left(xy+yz+zx+\lambda(x+y+z)\right)^3}$$

$$\ge\frac{(x+y+z)^4}{\left(\frac{(\sum x)^2}{3}+\lambda(x+y+z)\right)^3}=\frac{3}{(\lambda+1)^3}(as\ x+y+z=3)$$