ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0, a+b+c=1$ then:

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{27}{4}(ab+bc+ca)\geq\frac{45}{4}$$

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Solution by Tapas Das-India

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{27}{4}(ab+bc+ca)=$$

$$=\frac{3}{4}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)+\frac{1}{4}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)+\frac{27}{4}(ab+bc+ca)\overset{Bergstrom}{\geq}$$

$$\geq\frac{3}{4}\frac{(1+1+1)^2}{a+b+c}+\frac{1}{4}\frac{ab+bc+ca}{abc}+\frac{27}{4}(ab+bc+ca)\overset{AM-GM}{\geq}$$

$$\geq\frac{3}{4}.9+2\sqrt{\frac{27}{4}.\frac{1}{4}.\frac{1}{abc}.(ab+bc+ca)^2}\ (since\ a+b+c=1)\overset{(\sum x)^2\geq3\sum xy}{\geq}$$

$$\geq\frac{27}{4}+\frac{1}{2}\sqrt{\frac{27}{abc}.3abc(a+b+c)}=\frac{27}{4}+\frac{9}{2}=\frac{45}{4}$$

Equality holds for:

$$a=b=c=\frac{1}{3}$$