ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ such that : $abc = 1$ and $0 < \lambda \leq 27n$, then :

$$n(a+b+c)(ab+bc+ca) + \frac{\lambda}{ab+bc+ca} \geq 9n + \frac{\lambda}{3}$$

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$$\sum_{cyc} a \overset{A-G}{\geq} 3\sqrt[3]{abc} = 3 \to (1)$$

$$\therefore n(a+b+c)(ab+bc+ca) - 9n \geq$$

$$\geq 3n\sum_{cyc} ab - 9n = 3n\left(\sum_{cyc} ab - 3\right) \overset{?}{\geq} \frac{\lambda}{3} - \frac{\lambda}{ab+bc+ca}$$

$$\Leftrightarrow 3n\left(\sum_{cyc} ab - 3\right) \underset{(*)}{\overset{?}{\geq}} \lambda\left(\frac{\sum_{cyc} ab - 3}{3\sum_{cyc} ab}\right)$$

$$\text{Now, } 0 < \lambda \leq 27n \Rightarrow \text{RHS of } (*) \leq 27n\left(\frac{\sum_{cyc} ab - 3}{3\sum_{cyc} ab}\right)$$

$$\left(\because \sum_{cyc} ab \overset{A-G}{\geq} 3\sqrt[3]{a^2b^2c^2} \overset{abc=1}{=} 3 \Rightarrow \sum_{cyc} ab - 3 \geq 0\right) \overset{?}{\leq} 3n\left(\sum_{cyc} ab - 3\right)$$

$$\Leftrightarrow \sum_{cyc} ab - 3 - 3\left(\frac{\sum_{cyc} ab - 3}{\sum_{cyc} ab}\right) \overset{?}{\geq} 0 \; (\because n > 0) \Leftrightarrow \frac{1}{\sum_{cyc} ab}\left(\sum_{cyc} ab - 3\right)^2 \overset{?}{\geq} 0$$

$$\to \text{true} \Rightarrow (*) \text{ is true} \therefore n(a+b+c)(ab+bc+ca) - 9n \geq \frac{\lambda}{3} - \frac{\lambda}{ab+bc+ca}$$

$$\Rightarrow n(a+b+c)(ab+bc+ca) + \frac{\lambda}{ab+bc+ca} \geq 9n + \frac{\lambda}{3}$$

$$\forall\, a, b, c > 0 \,\middle|\, abc = 1 \text{ and } 0 < \lambda \leq 27n, \text{"="} \text{ iff } a = b = c = 1 \text{ (QED)}$$