ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0$ such that : $a+b+c=3$ and $n,k\in\mathbb{N}$ with $k+1\ge n$, then

$$\sum_{cyc}\frac{ab}{a^2+nb^2+ka}\le\frac{3}{n+k+1}$$

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$$a^2+nb^2+ka\overset{\text{weighted AM}\ge\text{weighted GM}}{\ge}(n+k+1).\sqrt[n+k+1]{a^2b^{2n}.a^k}$$

$$\Rightarrow\frac{ab}{a^2+nb^2+ka}\le\frac{1}{n+k+1}.\frac{a}{a^{\frac{k+2}{n+k+1}}}.\frac{b}{a^{\frac{2n}{n+k+1}}}=\frac{1}{n+k+1}.\left(a^{n-1}.b^{k+1-n}\right)^{\frac{1}{n+k+1}}$$

$$=\frac{1}{n+k+1}.\left(\sqrt[k]{a^{n-1}.b^{k+1-n}}\right)^{\frac{k}{n+k+1}}\overset{\substack{\text{weighted GM}\ge\text{weighted AM}\\ \text{since }k+1-n\ge0}}{\le}$$

$$\frac{1}{n+k+1}.\left(\frac{(n-1)a+(k+1-n)b}{k}\right)^{\frac{k}{n+k+1}}\overset{\text{Bernoulli}}{\le}$$

$$\frac{1}{n+k+1}.\left(1+\left(\frac{(n-1)a+(k+1-n)b-k}{k}\right).\frac{k}{n+k+1}\right)$$

$$\therefore\frac{ab}{a^2+nb^2+ka}\le\frac{1}{n+k+1}.\left(1+\frac{(n-1)(a-b)+bk-k}{n+k+1}\right)\text{ and analogs}$$

$$\Rightarrow\sum_{cyc}\frac{ab}{a^2+nb^2+ka}\le\frac{3}{n+k+1}$$

$$+\frac{(n-1)(a-b+b-c+c-a)+k(a+b+c)-3k}{(n+k+1)^2}$$

$$\overset{a+b+c=3}{=}\frac{3}{n+k+1}+\frac{3k-3k}{(n+k+1)^2}\therefore\sum_{cyc}\frac{ab}{a^2+nb^2+ka}\le\frac{3}{n+k+1}$$

$$\forall\, a,b,c>0\,\middle|\,a+b+c=3\text{ and }n,k\in\mathbb{N}\text{ with }k+1\ge n,$$

$$''=''\text{ iff }a=b=c=1\text{ (QED)}$$