ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0.\ a^2 + b^2 + c^2 = 3$ and $\lambda \geq 0$, then :

$$\sum_{cyc} \frac{a+\lambda}{a + bc + c^2} \geq ab + bc + ca + \lambda - 2$$

*Proposed by Marin Chirciu-Romania*

*Solution by Soumava Chakraborty-Kolkata-India*

$$3 = \sum_{cyc} a^2 \overset{A-G}{\geq} 3\sqrt[2]{a^2b^2c^2} \Rightarrow abc \leq 1 \to (1) \text{ and } 3 = \sum_{cyc} a^2 \geq \frac{1}{3}\left(\sum_{cyc} a\right)^2$$

$$\Rightarrow \sum_{cyc} a \leq 3 \to (2)$$

$$\text{Now, } \sum_{cyc} \frac{a}{a + bc + c^2}$$

$$= \sum_{cyc} \frac{a^2}{a^2 + abc + ac^2} \overset{\text{Bergstrom}}{\geq} \frac{\left(\sum_{cyc} a\right)^2}{\sum_{cyc} a^2 + 3abc + \sum_{cyc} a^2b} \overset{A-G}{\geq}$$

$$\frac{\left(\sum_{cyc} a\right)^2}{\sum_{cyc} a^2 + 3abc + \sum_{cyc} a^3} = \frac{\left(\sum_{cyc} a\right)^2}{\sum_{cyc} a^2 + 6abc + \left(\sum_{cyc} a\right)\left(\sum_{cyc} a^2 - \sum_{cyc} ab\right)}$$

$$\overset{\text{via (1) and (2)}}{\geq} \frac{\left(\sum_{cyc} a\right)^2}{\sum_{cyc} a^2 + 6 + 3\left(\sum_{cyc} a^2 - \sum_{cyc} ab\right)} \overset{3 = a^2+b^2+c^2}{=}$$

$$\frac{\left(\sum_{cyc} a\right)^2}{\sum_{cyc} a^2 + 2\sum_{cyc} a^2 + 3\left(\sum_{cyc} a^2 - \sum_{cyc} ab\right)} = \frac{\sum_{cyc} a^2 + 2\sum_{cyc} ab}{6\sum_{cyc} a^2 - 3\sum_{cyc} ab}$$

$$\overset{?}{\geq} ab + bc + ca - 2 = \frac{3\sum_{cyc} ab}{\sum_{cyc} a^2} - 2 \Leftrightarrow \frac{x + 2y}{6x - 3y} + 2 \overset{?}{\geq} \frac{3y}{x}$$

$$\left(x = \sum_{cyc} a^2 \text{ and } y = \sum_{cyc} ab\right) \Leftrightarrow \frac{13x - 4y}{6x - 3y} \overset{?}{\geq} \frac{3y}{x} \Leftrightarrow 13x^2 - 22xy + 9y^2 \overset{?}{\geq} 0$$

$$\Leftrightarrow (13x - 9y)(x - y) \overset{?}{\geq} 0 \to \text{true} \because x \geq y$$

$$\therefore \sum_{cyc} \frac{a}{a + bc + c^2} \geq ab + bc + ca - 2 \to \text{(i)}$$

$$\text{Again, } \sum_{cyc} \frac{1}{a + bc + c^2} \overset{\text{Bergstrom}}{\geq} \frac{9}{\sum_{cyc} a + \sum_{cyc} ab + \sum_{cyc} a^2} \overset{\text{via (2)}}{\geq} \frac{9}{3 + 2\sum_{cyc} a^2}$$

$$\overset{a^2+b^2+c^2=3}{=} \frac{9}{3+6} \Rightarrow \sum_{cyc} \frac{1}{a+bc+c^2} \geq 1 \text{ and } \because \lambda \geq 0 \therefore \sum_{cyc} \frac{\lambda}{a+bc+c^2} \geq \lambda$$

$$\to (ii)$$

$$\therefore (i)+(ii) \Rightarrow \sum_{cyc} \frac{a+\lambda}{a+bc+c^2} \geq ab+bc+ca+\lambda-2$$

$$\forall\, a,b,c > 0 \,\big|\, a^2+b^2+c^2 = 3 \text{ and } \forall\, \lambda \geq 0, '' = '' \text{ iff } a=b=c=1 \text{ (QED)}$$