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If $a, \mathrm{b}, \mathrm{c} > 0$ such that : $abc = 1$ and $\lambda \leq 3$, then :

$$(a+\mathrm{b})(\mathrm{b}+\mathrm{c})(\mathrm{c}+a) - \lambda(a+\mathrm{b}+\mathrm{c}) \geq 8 - 3\lambda$$

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$$(a+\mathrm{b})(\mathrm{b}+\mathrm{c})(\mathrm{c}+a) - \lambda(a+\mathrm{b}+\mathrm{c}) \geq 8 - 3\lambda$$

$$\Leftrightarrow \prod_{\mathrm{cyc}}(\mathrm{b}+\mathrm{c}) - 8 \geq \lambda\left(\sum_{\mathrm{cyc}} a - 3\right) \overset{abc=1}{\Leftrightarrow} \frac{\prod_{\mathrm{cyc}}(\mathrm{b}+\mathrm{c}) - 8abc}{abc} \overset{(*)}{\geq} \lambda\left(\sum_{\mathrm{cyc}} a - 3\right)$$

$$\text{Now, } \sum_{\mathrm{cyc}} a - 3 \overset{A-G}{\geq} 3\sqrt[3]{abc} - 3 \overset{abc=1}{=} 3 - 3 = 0 \Rightarrow \sum_{\mathrm{cyc}} a - 3 \geq 0$$

$$\Rightarrow (\lambda - 3)\left(\sum_{\mathrm{cyc}} a - 3\right) \leq 0 \ (\because \lambda - 3 \leq 0) \Rightarrow \lambda\left(\sum_{\mathrm{cyc}} a - 3\right) \overset{(\mathrm{i})}{\leq} 3\left(\sum_{\mathrm{cyc}} a - 3\right)$$

$\therefore$ (i) $\Rightarrow$ in order to prove $(*)$, it suffices to prove :

$$\frac{\prod_{\mathrm{cyc}}(\mathrm{b}+\mathrm{c}) - 8abc}{abc} \geq 3\left(\sum_{\mathrm{cyc}} a - 3\right) \overset{abc=1}{\Leftrightarrow} \frac{\prod_{\mathrm{cyc}}(\mathrm{b}+\mathrm{c}) - 8abc}{abc} + 9 \geq \frac{3\sum_{\mathrm{cyc}} a}{\sqrt[3]{abc}}$$

$$\Leftrightarrow \frac{\prod_{\mathrm{cyc}}(\mathrm{b}+\mathrm{c}) + abc}{abc} \overset{(**)}{\geq} \frac{3\sum_{\mathrm{cyc}} a}{\sqrt[3]{abc}}$$

Assigning $\mathrm{b}+\mathrm{c} = x, \mathrm{c}+a = \mathrm{y}, a+\mathrm{b} = \mathrm{z} \Rightarrow x+\mathrm{y}-\mathrm{z} = 2\mathrm{c} > 0, \mathrm{y}+\mathrm{z}-x = 2a > 0$ and $\mathrm{z}+x-\mathrm{y} = 2\mathrm{b} > 0 \Rightarrow x+\mathrm{y} > \mathrm{z}, \mathrm{y}+\mathrm{z} > x, \mathrm{z}+x > \mathrm{y} \Rightarrow x, \mathrm{y}, \mathrm{z}$ form sides of a triangle with semiperimeter, circumradius and inradius $= \mathrm{s}, \mathrm{R}, \mathrm{r}$ (say);

$$\text{so } 2\sum_{\mathrm{cyc}} a = \sum_{\mathrm{cyc}} x = 2\mathrm{s} \Rightarrow \sum_{\mathrm{cyc}} a = \mathrm{s} \rightarrow (1) \Rightarrow a = \mathrm{s}-x, \mathrm{b} = \mathrm{s}-\mathrm{y}, \mathrm{c} = \mathrm{s}-\mathrm{z}$$

$$\therefore abc = \mathrm{r}^2\mathrm{s} \rightarrow (2) \text{ and } (1) \text{ and } (2) \Rightarrow (**) \Leftrightarrow \frac{xyz + \mathrm{r}^2\mathrm{s}}{\mathrm{r}^2\mathrm{s}} \geq \frac{3\mathrm{s}}{\sqrt[3]{\mathrm{r}^2\mathrm{s}}}$$

$$\Leftrightarrow \frac{4\mathrm{Rrs} + \mathrm{r}^2\mathrm{s}}{\mathrm{r}^2\mathrm{s}} \geq \frac{3\mathrm{s}}{\sqrt[3]{\mathrm{r}^2\mathrm{s}}} \Leftrightarrow \frac{(4\mathrm{R}+\mathrm{r})^3}{\mathrm{r}^3} \geq \frac{27\mathrm{s}^3}{\mathrm{r}^2\mathrm{s}} \Leftrightarrow (4\mathrm{R}+\mathrm{r})^3 \overset{(***)}{\geq} 27\mathrm{rs}^2$$

$$\text{Now, } 27\mathrm{rs}^2 \overset{\text{Gerretsen}}{\leq} 27\mathrm{r}(4\mathrm{R}^2 + 4\mathrm{Rr} + 3\mathrm{r}^2) \overset{?}{\leq} (4\mathrm{R}+\mathrm{r})^3$$

$$\Leftrightarrow 16\mathrm{t}^3 - 15\mathrm{t}^2 - 24\mathrm{t} - 20 \overset{?}{\geq} 0 \ \left(\mathrm{t} = \frac{\mathrm{R}}{\mathrm{r}}\right) \Leftrightarrow (\mathrm{t}-2)(16\mathrm{t}^2 + 17\mathrm{t} + 10) \overset{?}{\geq} 0 \rightarrow \text{true}$$

$$\because \mathrm{t} \overset{\text{Euler}}{\geq} 2 \Rightarrow (***) \Rightarrow (**) \Rightarrow (*) \text{ is true}$$

$$\therefore (a+\mathrm{b})(\mathrm{b}+\mathrm{c})(\mathrm{c}+a) - \lambda(a+\mathrm{b}+\mathrm{c}) \geq 8 - 3\lambda$$

$$\forall\, a, \mathrm{b}, \mathrm{c} > 0 \,\middle|\, abc = 1, '' ='' \text{ iff } a = \mathrm{b} = \mathrm{c} = 1 \text{ (QED)}$$