ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0, and\ a+b+c=1, \lambda\geq\frac{7}{9}$ then:

$$\frac{1}{\lambda+a^2}+\frac{1}{\lambda+b^2}+\frac{1}{\lambda+c^2}\leq\frac{27}{9\lambda+1}$$

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Lemma: For $x\in\left(0,\frac{1}{3}\right)$ and $\lambda\geq\frac{7}{9}$:

$$\frac{1}{\lambda+x^2}\leq\frac{9}{9\lambda+1}+\frac{18-54x}{(9\lambda+1)^2}$$

Proof:

$$\frac{1}{\lambda+x^2}\leq\frac{9}{9\lambda+1}+\frac{18-54x}{(9\lambda+1)^2}\ or,$$

$$(9\lambda+1)^2\leq(27+81\lambda-54x)(\lambda+x^2)or,$$

$$54x^3-x^2(27+81\lambda)+54x\lambda-9\lambda+1\leq0\ or,$$

$$2p^3-p^2(3+9\lambda)+18p\lambda-9\lambda+1\overset{p=3x>0}{\leq}0\ or$$

$$(p-1)^2(2p+1-9\lambda)\leq0\ True.$$

$$Since\ x<\frac{1}{3}\ so\ p=3x<1\ and\ 2p+1<3\ and$$

$$9\lambda\geq9.\frac{7}{9}=7\ clearly\ (2p+1-9\lambda)<0$$

$$\frac{1}{\lambda+a^2}+\frac{1}{\lambda+b^2}+\frac{1}{\lambda+c^2}\overset{lemma}{\leq}\sum\left[\frac{9}{9\lambda+1}+\frac{18-54a}{(9\lambda+1)^2}\right]=$$

$$=\frac{27}{9\lambda+1}+\frac{54-54\sum a}{(9\lambda+1)^2}=\frac{27}{9\lambda+1}, as\ \sum a=1$$

$$Equality\ for:\ a=b=c=\frac{1}{3}$$