ROMANIAN MATHEMATICAL MAGAZINE

If $x,y,z>0,\ xy+yz+zx=3\ ,\ \lambda\ge 2$ then:

$$\frac{1}{x^2+\lambda}+\frac{1}{y^2+\lambda}+\frac{1}{z^2+\lambda}\le\frac{3}{\lambda+1}$$

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$$x^2+y^2+z^2\ge\ xy+yz+zx=3$$

$$(x+y+z)^2=x^2+y^2+z^2+2(xy+yz+zx)=\sum x^2+6$$

$$\sum\frac{1}{x^2+\lambda}=\frac{1}{\lambda}\sum\left(1-\frac{x^2}{x^2+\lambda}\right)=\frac{3}{\lambda}-\frac{1}{\lambda}\sum\frac{x^2}{x^2+\lambda}\overset{CBS}{\le}$$

$$\le\ \frac{3}{\lambda}-\frac{1}{\lambda}\left(\frac{(x+y+z)^2}{x^2+y^2+z^2+3\lambda}\right)=$$

$$=\frac{3}{\lambda}-\frac{1}{\lambda}\left(\frac{x^2+y^2+z^2+6}{x^2+y^2+z^2+3\lambda}\right)\overset{p=\sum x^2}{=}\frac{3}{\lambda}-\frac{1}{\lambda}\left(\frac{p+6}{p+3\lambda}\right)=$$

$$\frac{2}{\lambda}+\frac{1}{\lambda}\left(1-\frac{p+6}{p+3\lambda}\right)=\frac{2}{\lambda}+\frac{1}{\lambda}\left(\frac{3\lambda-6}{p+3\lambda}\right)\overset{p=\sum x^2\ge 3}{\le}$$

$$\le\ \frac{2}{\lambda}+\frac{1}{\lambda}\left(\frac{3\lambda-6}{3+3\lambda}\right)=\frac{1}{\lambda}\frac{9\lambda}{3+3\lambda}=\frac{3}{\lambda+1}$$

$$Equality\ for\ a=b=c=1$$