ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0, a^2+b^2+c^2=3, \lambda\ge 2+\sqrt{3}$ then:

$$\sum_{cyc}\frac{1}{\lambda-a}\le\frac{3}{\lambda-1}$$

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$$Let\ p=a^2, q=b^2, r=c^2\ and\ a^2+b^2+c^2=3\ or\ p+q+r=3$$

$Lemma$:

$$For\ x\in(0,3) and\ \lambda\ge 2+\sqrt{3}, \frac{1}{\lambda-\sqrt{x}}\le\frac{2\lambda-3+x}{2(\lambda-1)^2}$$

$Proof$:

$$\frac{1}{\lambda-\sqrt{x}}\le\frac{2\lambda-3+x}{2(\lambda-1)^2}\ \ or,$$

$$2(\lambda-1)^2\le[2\lambda-3+x](\lambda-\sqrt{x})or,$$

$$\lambda-2+x\lambda-\sqrt{x}(2\lambda-3)-x^{\frac{3}{2}}\ge 0\ or$$

$$u^3-u^2\lambda+u(2\lambda-3)-\lambda+2\overset{x=u^2}{\le}0\ or,$$

$$(u-1)(u+2-\lambda)\le 0$$

$$True, since\ x\in(0,3) and\ u=\sqrt{x}<\sqrt{3}, u+2<2+\sqrt{3}$$
$$and\ \lambda\ge 2+\sqrt{3}, so\ (u+2-\lambda)<0$$

$$Now\sum\frac{1}{\lambda-a}=\sum\frac{1}{\lambda-\sqrt{a^2}}=\sum\frac{1}{\lambda-\sqrt{p}}\overset{Lemma}{\le}$$
$$\sum\frac{2\lambda-3+p}{2(\lambda-1)^2}=\frac{3(2\lambda-3)+\sum p}{2(\lambda-1)^2}=$$

$$=\frac{6\lambda-9+3}{2(\lambda-1)^2}\left(as\sum p=3\right)=\frac{3}{\lambda-1}$$

$$Equality\ for\ a=b=c=1$$