ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0, abc=1, \lambda\ge 2$ then:

$$\sum_{cyc}\frac{1}{a^2+\lambda}\le\frac{3}{\lambda+1}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\sum a^2\overset{AM-GM}{\ge}3(abc)^{\frac{2}{3}}=3\qquad(1),$$

$$since\ abc=1\ and$$

$$(a+b+c)^2=\sum a^2+2\sum ab\overset{AM-GM}{\ge}\sum a^2+6(abc)^{\frac{2}{3}}=\sum a^2+6\qquad(2)$$

$$\sum\frac{1}{a^2+\lambda}=\frac{1}{\lambda}\sum\left(1-\frac{a^2}{a^2+\lambda}\right)=\frac{1}{\lambda}\left[3-\sum\frac{a^2}{a^2+\lambda}\right]\overset{CBS}{\le}$$

$$\le\frac{1}{\lambda}\left[3-\frac{(a+b+c)^2}{\sum a^2+3\lambda}\right]\overset{(2)}{\le}\frac{1}{\lambda}\left[3-\frac{\sum a^2+6}{\sum a^2+3\lambda}\right]=$$

$$\le\frac{1}{\lambda}\left[2+1-\frac{\sum a^2+6}{\sum a^2+3\lambda}\right]=\frac{1}{\lambda}\left(2+\frac{3\lambda-6}{\sum a^2+3\lambda}\right)\overset{(1)}{\le}$$

$$\le\frac{1}{\lambda}\left(2+\frac{3\lambda-6}{3+3\lambda}\right)=\frac{1}{\lambda}\frac{9\lambda}{(3+3\lambda)}=\frac{3}{\lambda+1}$$

$$Equality\ for\ a=b=c=1$$