

ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0, \lambda \geq 2$ then:

$$\sum_{cyc} \frac{a}{\lambda a + b} \leq \frac{3}{\lambda + 1}$$

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Solution by Tapas Das-India

$$\begin{aligned} \sum \frac{a}{\lambda a + b} &= \frac{1}{\lambda} \sum \frac{a\lambda}{\lambda a + b} = \frac{1}{\lambda} \sum \left(1 - \frac{b}{\lambda a + b}\right) = \\ &= \frac{3}{\lambda} - \frac{1}{\lambda} \sum \frac{b^2}{\lambda ab + b^2} \stackrel{CBS}{\leq} \frac{3}{\lambda} - \frac{1}{\lambda} \frac{(a+b+c)^2}{(\sum \lambda ab) + \sum b^2} = \\ &= \frac{3}{\lambda} - \frac{1}{\lambda} \frac{\sum a^2 + 2\sum ab}{(\sum \lambda ab) + \sum b^2} = \frac{2}{\lambda} + \frac{1}{\lambda} - \frac{1}{\lambda} \frac{\sum a^2 + 2\sum ab}{(\sum \lambda ab) + \sum b^2} = \\ &= \frac{2}{\lambda} + \frac{1}{\lambda} \frac{((\lambda - 2)\sum ab)}{\sum \lambda ab + \sum b^2} \stackrel{\sum a^2 \geq \sum ab}{\leq} \frac{2}{\lambda} + \frac{1}{\lambda} \frac{((\lambda - 2)\sum ab)}{\sum \lambda ab + \sum ab} = \\ &= \frac{2}{\lambda} + \frac{1}{\lambda} \frac{((\lambda - 2)\sum ab)}{(\lambda + 1)\sum ab} = \frac{2}{\lambda} + \frac{1}{\lambda} \frac{(\lambda - 2)}{(\lambda + 1)} = \frac{3}{\lambda + 1} \end{aligned}$$

Equality for $a = b = c = 1$