ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0, ab + bc + ca = abc$ and $3 \le \lambda \le 4$, then :

$$a + b + c \ge \lambda\left(\frac{a}{bc} + \frac{b}{ca} + \frac{c}{ab} - 1\right) + 9$$

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Assigning $b + c = x, c + a = y, a + b = z \Rightarrow x + y - z = 2c > 0, y + z - x = 2a > 0$ and $z + x - y = 2b > 0 \Rightarrow x + y > z, y + z > x, z + x > y \Rightarrow x, y, z$ form sides of a triangle with semiperimeter, circumradius and inradius $= s, R, r$ (say) yielding $2\sum_{cyc} a = \sum_{cyc} x = 2s \Rightarrow \sum_{cyc} a = s \rightarrow (1) \Rightarrow a = s - x, b = s - y, c = s - z$

$\therefore abc = r^2 s \rightarrow (2)$ and such substitutions $\Rightarrow \sum_{cyc} ab = \sum_{cyc}(s - x)(s - y)$

$$\Rightarrow \sum_{cyc} ab = 4Rr + r^2 \rightarrow (3), \sum_{cyc} a^2 = \left(\sum_{cyc} a\right)^2 - 2\sum_{cyc} ab \overset{\text{via (1) and (3)}}{=}$$

$$s^2 - 2(4Rr + r^2) \Rightarrow \sum_{cyc} a^2 = s^2 - 8Rr - 2r^2 \rightarrow (4)$$

$$\text{We have : } a + b + c - \lambda\left(\frac{a}{bc} + \frac{b}{ca} + \frac{c}{ab} - 1\right) - 9$$

$$= \frac{(\sum_{cyc} a)(\sum_{cyc} ab)}{abc} - \lambda\left(\frac{\sum_{cyc} a^2 - \sum_{cyc} ab}{\sum_{cyc} ab}\right) - 9 \left(\because \sum_{cyc} ab = abc\right)$$

$$\overset{\text{via (1),(2),(3) and (4)}}{=} \frac{s(4Rr + r^2)}{r^2 s} - \lambda\left(\frac{s^2 - 8Rr - 2r^2 - 4Rr - r^2}{4Rr + r^2}\right) - 9$$

$$= \frac{4R + r}{r} - \lambda\left(\frac{s^2 - 12Rr - 3r^2}{4Rr + r^2}\right) - 9 \overset{0 \le \lambda \le 4}{\ge} \frac{4R + r}{r} - 4\left(\frac{s^2 - 12Rr - 3r^2}{4Rr + r^2}\right) - 9$$

$$\left(\because s^2 - 12Rr - 3r^2 = s^2 - 16Rr + 5r^2 + 4r(R - 2r) \overset{\text{Gerretsen and Euler}}{\ge} 0\right)$$

$$= \frac{(4R + r)^2 - 4(s^2 - 12Rr - 3r^2) - 36Rr - 9r^2}{4Rr + r^2} \overset{\text{Gerretsen}}{\ge}$$

$$\frac{(4R + r)^2 - 4(4R^2 + 4Rr + 3r^2 - 12Rr - 3r^2) - 36Rr - 9r^2}{4Rr + r^2} = \frac{4r(R - 2r)}{4Rr + r^2} \overset{\text{Euler}}{\ge} 0$$

$$\therefore a + b + c \ge \lambda\left(\frac{a}{bc} + \frac{b}{ca} + \frac{c}{ab} - 1\right) + 9 \ \forall\, a, b, c > 0, \sum_{cyc} ab = abc \text{ and } 3 \le \lambda \le 4,$$

$$'' ='' \text{ iff } a = b = c = 3 \text{ (QED)}$$