ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0, abc = 1$, $\lambda \leq \frac{81}{2}$ then:

$$(a+b+c)(ab+bc+ca)+\frac{\lambda}{ab+bc+ca} \geq 9+\frac{\lambda}{3}$$

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Solution by Tapas Das-India

$$Let\ (ab+bc+ca)=x\ then \sum a=\sqrt{\left(\sum a\right)^2} \geq \sqrt{3\sum ab}=\sqrt{3x}$$

$$Now\ we\ need\ to\ show:$$

$$(a+b+c)(ab+bc+ca)+\frac{\lambda}{ab+bc+ca} \geq 9+\frac{\lambda}{3}\ or,$$

$$x\sqrt{3x}+\frac{\lambda}{x} \geq 9+\frac{\lambda}{3}\ or,$$

$$\sqrt{3}u^3+\frac{\lambda}{u^2} \overset{x=u^2>0}{\geq} 9+\frac{\lambda}{3} or, 3\sqrt{3}u^5-u^2(27+\lambda)+3\lambda \geq 0\ or,$$

$$\left(u-\sqrt{3}\right)\left(3\sqrt{3}u^2\left(u^2+\sqrt{3}u+3\right)-\left(u+\sqrt{3}\right)\lambda\right) \geq 0\ or,$$

$$\left(u-\sqrt{3}\right)\left(3\sqrt{3}u^2\left(u^2+\sqrt{3}u+3\right)-\left(u+\sqrt{3}\right)\frac{81}{2}\right) \overset{\lambda\leq\frac{81}{2}}{\geq} 0\ or,$$

$$\left(u-\sqrt{3}\right)^2\left(6\sqrt{3}u^3+36u^2+54\sqrt{3}u+81\right) \geq 0(True)$$

$$Equality\ for\ u=\sqrt{3}\ or\ u^2=x=ab+bc+ca=3\ or\ a=b=c=1$$