ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0$ and $\lambda\ge 0$, then :

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\ge\sum_{cyc}\frac{a^2+\lambda b^2}{a^3+\lambda b^3}$$

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$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-\sum_{cyc}\frac{a^2+\lambda b^2}{a^3+\lambda b^3}=\sum_{cyc}\left(\frac{1}{2a}+\frac{1}{2b}-\frac{a^2+\lambda b^2}{a^3+\lambda b^3}\right)$$

$$=\sum_{cyc}\frac{(b+a)(a^3+\lambda b^3)-2ab(a^2+\lambda b^2)}{2ab(a^3+\lambda b^3)}=\sum_{cyc}\frac{\lambda b^4-\lambda ab^3+a^4-a^3b}{2ab(a^3+\lambda b^3)}$$

$$=\sum_{cyc}\frac{\lambda b^3(b-a)-a^3(b-a)}{2ab(a^3+\lambda b^3)}=\sum_{cyc}\frac{(b-a)(\lambda b^3-a^3)}{2ab(a^3+\lambda b^3)}$$

$$=\sum_{cyc}\frac{(b-a)(\lambda b^3+a^3-2a^3)}{2ab(a^3+\lambda b^3)}=\sum_{cyc}\frac{b-a}{2ab}-\frac{a^3(b-a)}{ab(a^3+\lambda b^3)}$$

$$=\frac{1}{2abc}.\sum_{cyc}c(b-a)-\sum_{cyc}\frac{a^3c(b^3+\lambda c^3)(c^3+\lambda a^3)(b-a)}{abc(a^3+\lambda b^3)(b^3+\lambda c^3)(c^3+\lambda a^3)}=0-$$

$$\frac{1}{abc(a^3+\lambda b^3)(b^3+\lambda c^3)(c^3+\lambda a^3)}.\sum_{cyc}\left(a^3c\begin{pmatrix}b^3c^3+\lambda(a^3b^3+c^6)\\+\lambda^2c^3a^3\end{pmatrix}(b-a)\right)$$

$$=-\frac{a^3b^3c^3\sum_{cyc}c(b-a)+\lambda.\sum_{cyc}\left(a^3c(a^3b^3+c^6)(b-a)\right)+\lambda^2.\sum_{cyc}\left(a^6c^4(b-a)\right)}{abc(a^3+\lambda b^3)(b^3+\lambda c^3)(c^3+\lambda a^3)}.$$

$$=\frac{\lambda\left(\sum_{cyc}a^7b^4-abc(\sum_{cyc}a^5b^3)\right)+\lambda^2\left(\sum_{cyc}a^4b^7-abc(\sum_{cyc}a^3b^5)\right)}{abc(a^3+\lambda b^3)(b^3+\lambda c^3)(c^3+\lambda a^3)}$$

$$\therefore\text{ since }\lambda\ge 0\ \therefore\text{ in order to prove : }\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\ge\sum_{cyc}\frac{a^2+\lambda b^2}{a^3+\lambda b^3},$$

it sufficess to prove :

$$\boxed{\sum_{cyc}a^7b^4\overset{(*)}{\ge}abc\left(\sum_{cyc}a^5b^3\right)}\text{ and }\boxed{\sum_{cyc}a^4b^7\overset{(\bullet)}{\ge}abc\left(\sum_{cyc}a^3b^5\right)}$$

$$\text{Now, }\sum_{cyc}a^7b^4=abc\sum_{cyc}\frac{a^6b^3}{c}=abc\sum_{cyc}\frac{a^{10}b^6}{a^4b^3c}\overset{\text{Bergstrom}}{\ge}$$

$$abc.\frac{(\sum_{cyc}a^5b^3)^2}{abc(\sum_{cyc}a^3b^2)}\overset{?}{\ge}abc\left(\sum_{cyc}a^5b^3\right)\Leftrightarrow\sum_{cyc}a^5b^3\underset{(**)}{\overset{?}{\ge}}abc\left(\sum_{cyc}a^3b^2\right)\text{ and,}$$

$$\sum_{cyc}a^5b^3=abc\sum_{cyc}\frac{a^4b^2}{c}=abc\sum_{cyc}\frac{a^6b^4}{a^2b^2c}\overset{\text{Bergstrom}}{\ge}$$

$$abc.\frac{\left(\sum_{cyc}a^3b^2\right)^2}{abc\left(\sum_{cyc}ab\right)}\overset{?}{\geq}abc\left(\sum_{cyc}a^3b^2\right)\Leftrightarrow\sum_{cyc}a^3b^2\underset{(***)}{\overset{?}{\geq}}abc\left(\sum_{cyc}ab\right)\text{ and,}$$

$$\sum_{cyc}a^3b^2=abc\sum_{cyc}\frac{a^2b}{c}=abc\sum_{cyc}\frac{a^2b^2}{bc}\overset{\text{Bergstrom}}{\geq}abc.\frac{\left(\sum_{cyc}ab\right)^2}{\sum_{cyc}ab}=abc\left(\sum_{cyc}ab\right)$$

$$\Rightarrow(***)\Rightarrow(**)\Rightarrow\boxed{(*)\text{ is true}}$$

$$\text{Again,}\sum_{cyc}a^4b^7=abc\sum_{cyc}\frac{a^3b^6}{c}=abc\sum_{cyc}\frac{a^6b^{10}}{a^3b^4c}\overset{\text{Bergstrom}}{\geq}$$

$$abc.\frac{\left(\sum_{cyc}a^3b^5\right)^2}{abc\left(\sum_{cyc}a^2b^3\right)}\overset{?}{\geq}abc\left(\sum_{cyc}a^3b^5\right)\Leftrightarrow\sum_{cyc}a^3b^5\underset{(\bullet\bullet)}{\overset{?}{\geq}}abc\left(\sum_{cyc}a^2b^3\right)\text{ and,}$$

$$\sum_{cyc}a^3b^5=abc\sum_{cyc}\frac{a^2b^4}{c}=abc\sum_{cyc}\frac{a^4b^6}{a^2b^2c}\overset{\text{Bergstrom}}{\geq}$$

$$abc.\frac{\left(\sum_{cyc}a^2b^3\right)^2}{abc\left(\sum_{cyc}ab\right)}\overset{?}{\geq}abc\left(\sum_{cyc}a^2b^3\right)\Leftrightarrow\sum_{cyc}a^2b^3\underset{(\bullet\bullet\bullet)}{\overset{?}{\geq}}abc\left(\sum_{cyc}ab\right)\text{ and,}$$

$$\sum_{cyc}a^2b^3=abc\sum_{cyc}\frac{ab^2}{c}=abc\sum_{cyc}\frac{a^2b^2}{ca}\overset{\text{Bergstrom}}{\geq}abc.\frac{\left(\sum_{cyc}ab\right)^2}{\sum_{cyc}ab}=abc\left(\sum_{cyc}ab\right)$$

$$\Rightarrow(\bullet\bullet\bullet)\Rightarrow(\bullet\bullet)\Rightarrow\boxed{(\bullet)\text{ is true}}\therefore(*)\text{ and }(\bullet)\text{ are both true}$$

$$\therefore\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\geq\sum_{cyc}\frac{a^2+\lambda b^2}{a^3+\lambda b^3}\ \forall\, a,b,c>0\text{ and }\lambda\geq 0,''='' \text{ iff } a=b=c\ (\text{QED})$$