ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ and $\lambda \geq 0$, then :

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geq \sum_{cyc} \frac{a^3+\lambda b^3}{a^4+\lambda b^4}$$

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$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-\sum_{cyc}\frac{a^3+\lambda b^3}{a^4+\lambda b^4}=\sum_{cyc}\left(\frac{1}{2a}+\frac{1}{2b}-\frac{a^3+\lambda b^3}{a^4+\lambda b^4}\right)$$

$$=\sum_{cyc}\frac{(b+a)(a^4+\lambda b^4)-2ab(a^3+\lambda b^3)}{2ab(a^4+\lambda b^4)}=\sum_{cyc}\frac{\lambda b^5-\lambda ab^4+a^5-a^4b}{2ab(a^4+\lambda b^4)}$$

$$=\sum_{cyc}\frac{\lambda b^4(b-a)-a^4(b-a)}{2ab(a^4+\lambda b^4)}=\sum_{cyc}\frac{(b-a)(\lambda b^4-a^4)}{2ab(a^4+\lambda b^4)}$$

$$=\sum_{cyc}\frac{(b-a)(\lambda b^4+a^4-2a^4)}{2ab(a^4+\lambda b^4)}=\sum_{cyc}\frac{b-a}{2ab}-\sum_{cyc}\frac{a^4(b-a)}{ab(a^4+\lambda b^4)}$$

$$=\frac{1}{2abc}.\sum_{cyc}c(b-a)-\sum_{cyc}\frac{a^4c(b^4+\lambda c^4)(c^4+\lambda a^4)(b-a)}{abc(a^4+\lambda b^4)(b^4+\lambda c^4)(c^4+\lambda a^4)}$$

$$=0-\frac{\sum_{cyc}\left(a^4c\left(b^4c^4+\lambda(a^4b^4+c^8)+\lambda^2c^4a^4\right)(b-a)\right)}{abc(a^4+\lambda b^4)(b^4+\lambda c^4)(c^4+\lambda a^4)}$$

$$=-\frac{a^4b^4c^4\sum_{cyc}c(b-a)+\lambda.\sum_{cyc}\left(a^4c(a^4b^4+c^8)(b-a)\right)+\lambda^2.\sum_{cyc}\left(a^8c^5(b-a)\right)}{abc(a^4+\lambda b^4)(b^4+\lambda c^4)(c^4+\lambda a^4)}$$

$$=\frac{\lambda\left(\sum_{cyc}a^9b^5-abc\left(\sum_{cyc}a^7b^4\right)\right)+\lambda^2\left(\sum_{cyc}a^5b^9-abc\left(\sum_{cyc}a^4b^7\right)\right)}{abc(a^4+\lambda b^4)(b^4+\lambda c^4)(c^4+\lambda a^4)}$$

$\therefore$ since $\lambda \geq 0 \therefore$ in order to prove : $\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geq \sum_{cyc}\frac{a^3+\lambda b^3}{a^4+\lambda b^4}$, it suffices

to prove : $\boxed{\sum_{cyc}a^9b^5 \overset{(*)}{\geq} abc\left(\sum_{cyc}a^7b^4\right)}$ and $\boxed{\sum_{cyc}a^5b^9 \overset{(\bullet)}{\geq} abc\left(\sum_{cyc}a^4b^7\right)}$

$$\text{Now, }\sum_{cyc}a^9b^5=abc\sum_{cyc}\frac{a^8b^4}{c}=abc\sum_{cyc}\frac{a^{14}b^8}{a^6b^4c}\overset{\text{Bergstrom}}{\geq}abc.\frac{\left(\sum_{cyc}a^7b^4\right)^2}{abc\left(\sum_{cyc}a^5b^3\right)}$$

$$\overset{?}{\geq}abc\left(\sum_{cyc}a^7b^4\right)\Leftrightarrow\sum_{cyc}a^7b^4\underset{(**)}{\overset{?}{\geq}}abc\left(\sum_{cyc}a^5b^3\right)\text{ and,}$$

$$\sum_{cyc}a^7b^4=abc\sum_{cyc}\frac{a^6b^3}{c}=abc\sum_{cyc}\frac{a^{10}b^6}{a^4b^3c}\overset{\text{Bergstrom}}{\geq}abc.\frac{\left(\sum_{cyc}a^5b^3\right)^2}{abc\left(\sum_{cyc}a^3b^2\right)}$$

$$\overset{?}{\geq}abc\left(\sum_{cyc}a^5b^3\right)\Leftrightarrow\sum_{cyc}a^5b^3\underset{(***)}{\overset{?}{\geq}}abc\left(\sum_{cyc}a^3b^2\right)\text{ and,}$$

$$\sum_{cyc} a^5b^3 = abc\sum_{cyc}\frac{a^4b^2}{c} = abc\sum_{cyc}\frac{a^6b^4}{a^2b^2c} \overset{\text{Bergstrom}}{\geq} abc.\frac{\left(\sum_{cyc} a^3b^2\right)^2}{abc\left(\sum_{cyc} ab\right)}$$

$$\overset{?}{\geq} abc\left(\sum_{cyc} a^3b^2\right) \Leftrightarrow \sum_{cyc} a^3b^2 \underset{(****)}{\overset{?}{\geq}} abc\left(\sum_{cyc} ab\right) \text{ and,}$$

$$\sum_{cyc} a^3b^2 = abc\sum_{cyc}\frac{a^2b}{c} = abc\sum_{cyc}\frac{a^2b^2}{bc} \overset{\text{Bergstrom}}{\geq} abc.\frac{\left(\sum_{cyc} ab\right)^2}{\sum_{cyc} ab} = abc\left(\sum_{cyc} ab\right)$$

$$\Rightarrow (****) \Rightarrow (***) \Rightarrow (**) \Rightarrow \boxed{(*)\text{ is true}}$$

$$\text{Again, } \sum_{cyc} a^5b^9 = abc\sum_{cyc}\frac{a^4b^8}{c} = abc\sum_{cyc}\frac{a^8b^{14}}{a^4b^6c} \overset{\text{Bergstrom}}{\geq} abc.\frac{\left(\sum_{cyc} a^4b^7\right)^2}{abc\left(\sum_{cyc} a^3b^5\right)}$$

$$\overset{?}{\geq} abc\left(\sum_{cyc} a^4b^7\right) \Leftrightarrow \sum_{cyc} a^4b^7 \underset{(\bullet\bullet)}{\overset{?}{\geq}} abc\left(\sum_{cyc} a^3b^5\right) \text{ and,}$$

$$\sum_{cyc} a^4b^7 = abc\sum_{cyc}\frac{a^3b^6}{c} = abc\sum_{cyc}\frac{a^6b^{10}}{a^3b^4c} \overset{\text{Bergstrom}}{\geq} abc.\frac{\left(\sum_{cyc} a^3b^5\right)^2}{abc\left(\sum_{cyc} a^2b^3\right)}$$

$$\overset{?}{\geq} abc\left(\sum_{cyc} a^3b^5\right) \Leftrightarrow \sum_{cyc} a^3b^5 \underset{(\bullet\bullet\bullet)}{\overset{?}{\geq}} abc\left(\sum_{cyc} a^2b^3\right) \text{ and,}$$

$$\sum_{cyc} a^3b^5 = abc\sum_{cyc}\frac{a^2b^4}{c} = abc\sum_{cyc}\frac{a^4b^6}{a^2b^2c} \overset{\text{Bergstrom}}{\geq} abc.\frac{\left(\sum_{cyc} a^2b^3\right)^2}{abc\left(\sum_{cyc} ab\right)}$$

$$\overset{?}{\geq} abc\left(\sum_{cyc} a^2b^3\right) \Leftrightarrow \sum_{cyc} a^2b^3 \underset{(\bullet\bullet\bullet\bullet)}{\overset{?}{\geq}} abc\left(\sum_{cyc} ab\right) \text{ and,}$$

$$\sum_{cyc} a^2b^3 = abc\sum_{cyc}\frac{ab^2}{c} = abc\sum_{cyc}\frac{a^2b^2}{ca} \overset{\text{Bergstrom}}{\geq} abc.\frac{\left(\sum_{cyc} ab\right)^2}{\sum_{cyc} ab} = abc\left(\sum_{cyc} ab\right)$$

$$\Rightarrow (\bullet\bullet\bullet\bullet) \Rightarrow (\bullet\bullet\bullet) \Rightarrow (\bullet\bullet) \Rightarrow \boxed{(\bullet)\text{ is true}} \therefore (*) \text{ and } (\bullet) \text{ are both true}$$

$$\therefore \frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geq \sum_{cyc}\frac{a^3+\lambda b^3}{a^4+\lambda b^4} \forall\, a,b,c>0 \text{ and } \lambda \geq 0, \text{"} = \text{" iff } a=b=c \text{ (QED)}$$