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If $x,y,z>0$ and $x+y+z=3$ with $\lambda\geq 2$, then :

$$\sum_{cyc}\frac{x^2}{x+\lambda y^2}\geq\frac{3}{\lambda+1}$$

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$$\sum_{cyc}\frac{x^2}{x+\lambda y^2}=\sum_{cyc}\frac{x^4}{x^3+\lambda x^2y^2}\overset{\text{Bergstrom}}{\geq}\frac{\left(\sum_{cyc}x^2\right)^2}{\sum_{cyc}x^3+\lambda\sum_{cyc}x^2y^2}\overset{x+y+z=3}{=}$$

$$\frac{\left(\sum_{cyc}x^2\right)^2}{\frac{\left(\sum_{cyc}x^3\right)\left(\sum_{cyc}x\right)}{3}+\lambda\sum_{cyc}x^2y^2}=\frac{3\left(\sum_{cyc}x^2\right)^2}{\left(\sum_{cyc}x^3\right)\left(\sum_{cyc}x\right)+3\lambda\sum_{cyc}x^2y^2}\overset{?}{\geq}\frac{3}{\lambda+1}$$

$$\Leftrightarrow\lambda\left(\left(\sum_{cyc}x^2\right)^2-3\sum_{cyc}x^2y^2\right)+\left(\sum_{cyc}x^2\right)^2-\left(\sum_{cyc}x^3\right)\left(\sum_{cyc}x\right)\boxed{\overset{?}{\underset{(*)}{\geq}}}0$$

Assigning $y+z=a, z+x=b, x+y=c\Rightarrow a+b-c=2z>0, b+c-a=2x>0$ and $c+a-b=2y>0\Rightarrow a+b>c, b+c>a, c+a>b\Rightarrow a,b,c$ form sides of a triangle with semiperimeter, circumradius and inradius $=s,R,r$ (say)

yielding $2\sum_{cyc}x=\sum_{cyc}a=2s\Rightarrow\sum_{cyc}x\overset{(\bullet)}{=}s\Rightarrow x=s-a, y=s-b, z=s-c$

$\therefore xyz\overset{(\bullet\bullet)}{=}r^2s$ and, $\sum_{cyc}xy=\sum_{cyc}(s-a)(s-b)=4Rr+r^2\Rightarrow\sum_{cyc}xy\overset{(\bullet\bullet\bullet)}{=}4Rr+r^2$

and also, $\sum_{cyc}x^2=\left(\sum_{cyc}x\right)^2-2\sum_{cyc}xy\overset{\text{via }(\bullet)\text{ and }(\bullet\bullet\bullet)}{=}s^2-2(4Rr+r^2)\Rightarrow$

$\sum_{cyc}x^2\overset{(\bullet\bullet\bullet\bullet)}{=}s^2-8Rr-2r^2$ and also, $\sum_{cyc}x^2y^2=\left(\sum_{cyc}xy\right)^2-2xyz\left(\sum_{cyc}x\right)$

$\overset{\text{via }(\bullet),(\bullet\bullet)\text{ and }(\bullet\bullet\bullet)}{=}(4Rr+r^2)^2-2r^2s^2\rightarrow(\bullet\bullet\bullet\bullet\bullet)$ and $\sum_{cyc}x^3=$

$$\left(\sum_{cyc}x\right)^3-3(x+y)(y+z)(z+x)\overset{\text{via }(\bullet)}{=}s^3-12Rrs\rightarrow(\bullet\bullet\bullet\bullet\bullet\bullet)$$

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$$\therefore \text{ via } (\bullet\bullet\bullet\bullet), (\bullet\bullet\bullet\bullet\bullet) \text{ and } (\bullet\bullet\bullet\bullet\bullet\bullet) \text{ and } \because \lambda \geq 2 \text{ and } \left(\sum_{cyc} x^2\right)^2 - 3\sum_{cyc} x^2y^2 \geq 0,$$

$$\text{LHS of } (*) \geq 2\left((s^2-8Rr-2r^2)^2 - 3((4Rr+r^2)^2 - 2r^2s^2)\right) + (s^2-8Rr-2r^2)^2$$

$$-s(s^3-12Rrs) \overset{?}{\geq} 0 \Leftrightarrow s^4 - 18Rrs^2 + 3r^2(4R+r)^2 \boxed{\underset{(**)}{\overset{?}{\geq}}} 0 \text{ and}$$

$$\because (s^2-16Rr+5r^2)^2 \overset{\text{Gerretsen}}{\geq} 0 \therefore \text{in order to prove } (**), \text{it suffices to prove :}$$

$$\text{LHS of } (**) \geq (s^2-16Rr+5r^2)^2 \Leftrightarrow (7R-5r)s^2 \boxed{\underset{(***)}{\overset{?}{\geq}}} r(104R^2-92Rr+11r^2)$$

$$\text{Now, } (7R-5r)s^2 \overset{\text{Gerretsen}}{\geq} (7R-5r)(16Rr-5r^2) \overset{?}{\geq} r(104R^2-92Rr+11r^2)$$

$$\Leftrightarrow 8R^2-23Rr+14r^2 \overset{?}{\geq} 0 \Leftrightarrow (8R-7r)(R-2r) \overset{?}{\geq} 0 \rightarrow \text{true} \because R \overset{\text{Euler}}{\geq} 2r$$

$$\Rightarrow (***) \Rightarrow (**) \Rightarrow (*) \text{ is true } \therefore \sum_{cyc} \frac{x^2}{x+\lambda y^2} \geq \frac{3}{\lambda+1} \; \forall x,y,z > 0 \,\middle|\, x+y+z=3$$

$$\text{with } \lambda \geq 2, \text{"="} \text{ iff } x=y=z=1 \text{ (QED)}$$