ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ and $\lambda \geq \frac{1}{4}$, then :

$$\frac{a}{a+b}+\frac{b}{b+c}+\frac{c}{c+a}+\frac{\lambda(a^2+b^2+c^2)}{ab+bc+ca} \geq \lambda+\frac{3}{2}$$

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Solution by Soumava Chakraborty-Kolkata-India

$$\frac{a}{a+b}+\frac{b}{b+c}+\frac{c}{c+a}+\frac{\lambda(a^2+b^2+c^2)}{ab+bc+ca}-\lambda-\frac{3}{2}=$$

$$=\sum_{cyc}\frac{a^2}{a^2+ab}-\frac{3}{2}+\frac{\lambda\left(\sum_{cyc}a^2-\sum_{cyc}ab\right)}{\sum_{cyc}ab} \geq$$

$$\overset{\text{Bergstrom}}{\geq} \frac{\sum_{cyc}a^2+2\sum_{cyc}ab}{\sum_{cyc}a^2+\sum_{cyc}ab}-\frac{3}{2}+\frac{\lambda\left(\sum_{cyc}a^2-\sum_{cyc}ab\right)}{\sum_{cyc}ab} \overset{\lambda \geq \frac{1}{4}}{\geq}$$

$$\geq \frac{y-x}{2(x+y)}+\frac{x-y}{4y}\left(x=\sum_{cyc}a^2 \text{ and } y=\sum_{cyc}ab\right)=$$

$$=\left(\frac{x-y}{2}\right)\left(\frac{1}{2y}-\frac{1}{x+y}\right)=\frac{(x-y)^2}{4y(x+y)} \geq 0$$

$$\therefore \frac{a}{a+b}+\frac{b}{b+c}+\frac{c}{c+a}+\frac{\lambda(a^2+b^2+c^2)}{ab+bc+ca} \geq \lambda+\frac{3}{2} \; \forall\, a,b,c>0 \,\Big|\, \lambda \geq \frac{1}{4},$$
$$'' ='' \text{ iff } a=b=c \text{ (QED)}$$