ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0$ and $\lambda\ge 0$ with $n\in\mathbb{N}$, then :

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\ge\sum_{cyc}\frac{a^n+\lambda b^n}{a^{n+1}+\lambda b^{n+1}}$$

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$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-\sum_{cyc}\frac{a^n+\lambda b^n}{a^{n+1}+\lambda b^{n+1}}=\sum_{cyc}\left(\frac{1}{2a}+\frac{1}{2b}-\frac{a^n+\lambda b^n}{a^{n+1}+\lambda b^{n+1}}\right)$$

$$=\sum_{cyc}\frac{(b+a)\left(a^{n+1}+\lambda b^{n+1}\right)-2ab(a^n+\lambda b^n)}{2ab(a^{n+1}+\lambda b^{n+1})}$$

$$=\sum_{cyc}\frac{\lambda b^{n+2}-\lambda ab^{n+1}+a^{n+2}-a^{n+1}b}{2ab(a^{n+1}+\lambda b^{n+1})}$$

$$=\sum_{cyc}\frac{\lambda b^{n+1}(b-a)-a^{n+1}(b-a)}{2ab(a^{n+1}+\lambda b^{n+1})}=\sum_{cyc}\frac{(b-a)\left(\lambda b^{n+1}-a^{n+1}\right)}{2ab(a^{n+1}+\lambda b^{n+1})}$$

$$=\sum_{cyc}\frac{(b-a)\left(\lambda b^{n+1}+a^{n+1}-2a^{n+1}\right)}{2ab(a^{n+1}+\lambda b^{n+1})}=\sum_{cyc}\frac{b-a}{2ab}-\sum_{cyc}\frac{a^{n+1}(b-a)}{ab(a^{n+1}+\lambda b^{n+1})}$$

$$=\frac{1}{2abc}.\sum_{cyc}c(b-a)-\sum_{cyc}\frac{a^{n+1}c\left(b^{n+1}+\lambda c^{n+1}\right)\left(c^{n+1}+\lambda a^{n+1}\right)(b-a)}{abc(a^{n+1}+\lambda b^{n+1})(b^{n+1}+\lambda c^{n+1})(c^{n+1}+\lambda a^{n+1})}$$

$$=0-\frac{\sum_{cyc}\left(a^{n+1}c\left(b^{n+1}c^{n+1}+\lambda\left(a^{n+1}b^{n+1}+c^8\right)+\lambda^2c^{n+1}a^{n+1}\right)(b-a)\right)}{abc(a^{n+1}+\lambda b^{n+1})(b^{n+1}+\lambda c^{n+1})(c^{n+1}+\lambda a^{n+1})}$$

$$=-\frac{a^{n+1}b^{n+1}c^{n+1}\sum_{cyc}c(b-a)+\lambda.\sum_{cyc}\left(a^{n+1}c\left(a^{n+1}b^{n+1}+c^{2n+2}\right)(b-a)\right)}{abc(a^{n+1}+\lambda b^{n+1})(b^{n+1}+\lambda c^{n+1})(c^{n+1}+\lambda a^{n+1})}$$

$$-\frac{\lambda^2.\sum_{cyc}\left(a^{2n+2}c^{n+2}(b-a)\right)}{abc(a^{n+1}+\lambda b^{n+1})(b^{n+1}+\lambda c^{n+1})(c^{n+1}+\lambda a^{n+1})}$$

$$=\frac{\lambda\left(\sum_{cyc}a^{2n+3}b^{n+2}-abc\left(\sum_{cyc}a^{2n+1}b^{n+1}\right)\right)+\lambda^2\left(\sum_{cyc}a^{n+1}b^{2n+3}-abc\left(\sum_{cyc}a^{n+1}b^{2n+1}\right)\right)}{abc(a^{n+1}+\lambda b^{n+1})(b^{n+1}+\lambda c^{n+1})(c^{n+1}+\lambda a^{n+1})}$$

$\because \lambda\ge 0\ \therefore$ in order to prove : $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\ge\sum_{cyc}\frac{a^n+\lambda b^n}{a^{n+1}+\lambda b^{n+1}}$, it sufficess to prove :

$$\boxed{\sum_{cyc}a^{2n+3}b^{n+2}\overset{(*)}{\ge}abc\left(\sum_{cyc}a^{2n+1}b^{n+1}\right)}\ \text{and}\ \boxed{\sum_{cyc}a^{n+1}b^{2n+3}\overset{(\bullet)}{\ge}abc\left(\sum_{cyc}a^{n+1}b^{2n+1}\right)}$$

Let's prove $(*)$ for $n=0,1,2$ and indeed we have, $\sum_{cyc}a^3b^2=abc\sum_{cyc}\frac{a^2b}{c}$

$$=abc\sum_{cyc}\frac{a^2b^2}{bc}\overset{\text{Bergstrom}}{\ge}abc.\frac{\left(\sum_{cyc}ab\right)^2}{\sum_{cyc}ab}=abc\left(\sum_{cyc}ab\right)$$

$$\Rightarrow \sum_{cyc} a^3b^2 \geq abc\left(\sum_{cyc} ab\right) \to (1)$$

$$\text{and}, \sum_{cyc} a^5b^3 = abc\sum_{cyc}\frac{a^4b^2}{c} = abc\sum_{cyc}\frac{a^6b^4}{a^2b^2c} \overset{\text{Bergstrom}}{\geq} abc.\frac{\left(\sum_{cyc} a^3b^2\right)^2}{abc\left(\sum_{cyc} ab\right)}$$

$$\overset{\text{via }(1)}{\geq} \frac{abc\left(\sum_{cyc} ab\right)\left(\sum_{cyc} a^3b^2\right)}{\sum_{cyc} ab} \Rightarrow \sum_{cyc} a^5b^3 \geq abc\left(\sum_{cyc} a^3b^2\right) \to (2)$$

$$\text{and}, \sum_{cyc} a^7b^4 = abc\sum_{cyc}\frac{a^6b^3}{c} = abc\sum_{cyc}\frac{a^{10}b^6}{a^4b^3c} \overset{\text{Bergstrom}}{\geq} abc.\frac{\left(\sum_{cyc} a^5b^3\right)^2}{abc\left(\sum_{cyc} a^3b^2\right)}$$

$$\overset{\text{via }(2)}{\geq} \frac{abc\left(\sum_{cyc} a^3b^2\right)\left(\sum_{cyc} a^5b^3\right)}{\sum_{cyc} a^3b^2} \Rightarrow \sum_{cyc} a^7b^4 \geq abc\left(\sum_{cyc} a^5b^3\right) \to (3)$$

$\therefore (1),(2),(3) \Rightarrow \boxed{(*)\text{ is true for } n = 0, 1 \text{ and } 2}$ and we now assume $(*)$ is true for $0, 1, 2, \ldots, n$ and we shall subsequently prove that : $(*)$ is true for $n \equiv n+1$

$$\text{For } n \equiv n+1, (*) \Leftrightarrow \sum_{cyc} a^{2n+5}b^{n+3} \overset{(**)}{\geq} abc\left(\sum_{cyc} a^{2n+3}b^{n+2}\right) \text{ and } \sum_{cyc} a^{2n+5}b^{n+3} =$$

$$abc\sum_{cyc}\frac{a^{2n+4}b^{n+2}}{c} = abc\sum_{cyc}\frac{a^{4n+6}b^{2n+4}}{ca^{2n+2}b^{n+2}} \overset{\text{Bergstrom}}{\geq} \frac{abc\left(\sum_{cyc} a^{2n+3}b^{n+2}\right)^2}{abc\left(\sum_{cyc} a^{2n+1}b^{n+1}\right)}$$

$$\overset{\text{via }(*),\text{our assumption}}{\geq} \frac{abc\left(\sum_{cyc} a^{2n+1}b^{n+1}\right)\left(\sum_{cyc} a^{2n+3}b^{n+2}\right)}{\sum_{cyc} a^{2n+1}b^{n+1}} = abc\left(\sum_{cyc} a^{2n+3}b^{n+2}\right)$$

$\Rightarrow (**)$ is true whenever $(*)$ is true

$\Rightarrow$ via principle of mathematical induction, $\boxed{(*)\text{ is true } \forall\, n \in \mathbb{N}}$

Let's prove $(\bullet)$ for $n = 0, 1, 2$ and indeed we have, $\sum_{cyc} a^2b^3 = abc\sum_{cyc}\frac{ab^2}{c}$

$$= abc\sum_{cyc}\frac{a^2b^2}{ca} \overset{\text{Bergstrom}}{\geq} abc.\frac{\left(\sum_{cyc} ab\right)^2}{\sum_{cyc} ab} = abc\left(\sum_{cyc} ab\right)$$

$$\Rightarrow \sum_{cyc} a^2b^3 \geq abc\left(\sum_{cyc} ab\right) \to (4)$$

$$\text{and}, \sum_{cyc} a^3b^5 = abc\sum_{cyc}\frac{a^2b^4}{c} = abc\sum_{cyc}\frac{a^4b^6}{a^2b^2c} \overset{\text{Bergstrom}}{\geq} abc.\frac{\left(\sum_{cyc} a^2b^3\right)^2}{abc\left(\sum_{cyc} ab\right)}$$

$$\overset{\text{via }(4)}{\geq} \frac{abc\left(\sum_{cyc} ab\right)\left(\sum_{cyc} a^2b^3\right)}{\sum_{cyc} ab} \Rightarrow \sum_{cyc} a^3b^5 \geq abc\left(\sum_{cyc} a^2b^3\right) \to (5)$$

$$\text{and}, \sum_{cyc} a^4b^7 = abc\sum_{cyc}\frac{a^3b^6}{c} = abc\sum_{cyc}\frac{a^6b^{10}}{a^3b^4c} \overset{\text{Bergstrom}}{\geq} abc.\frac{\left(\sum_{cyc} a^3b^5\right)^2}{abc\left(\sum_{cyc} a^2b^3\right)}$$

$$\overset{\text{via }(5)}{\geq} \frac{abc\left(\sum_{cyc} a^2b^3\right)\left(\sum_{cyc} a^3b^5\right)}{\sum_{cyc} a^2b^3} \Rightarrow \sum_{cyc} a^4b^7 \geq abc\left(\sum_{cyc} a^3b^5\right) \rightarrow (6)$$

$\therefore$ $\boxed{(4),(5),(6) \Rightarrow (\bullet) \text{ is true for } n = 0, 1 \text{ and } 2}$ and we now assume $(\bullet)$ is true for $0, 1, 2, \ldots, n$ and we shall subsequently prove that : $(\bullet)$ is true for $n \equiv n+1$

$$\text{For } n \equiv n+1, (*) \Leftrightarrow \sum_{cyc} a^{n+3}b^{2n+5} \overset{(\bullet\bullet)}{\geq} abc\left(\sum_{cyc} a^{n+2}b^{2n+3}\right) \text{ and } \sum_{cyc} a^{n+3}b^{2n+5} =$$

$$abc\sum_{cyc}\frac{a^{n+2}b^{2n+4}}{c} = abc\sum_{cyc}\frac{a^{2n+4}b^{4n+6}}{ca^{n+2}b^{2n+2}} \overset{\text{Bergstrom}}{\geq} \frac{abc\left(\sum_{cyc} a^{n+2}b^{2n+3}\right)^2}{abc\left(\sum_{cyc} a^{n+1}b^{2n+1}\right)}$$

$$\overset{\text{via }(\bullet)\text{, our assumption}}{\geq} \frac{abc\left(\sum_{cyc} a^{n+1}b^{2n+1}\right)\left(\sum_{cyc} a^{n+2}b^{2n+3}\right)}{\sum_{cyc} a^{n+1}b^{2n+1}} = abc\left(\sum_{cyc} a^{n+2}b^{2n+3}\right)$$

$\Rightarrow (\bullet\bullet)$ is true whenever $(\bullet)$ is true

$\Rightarrow$ via principle of mathematical induction, $\boxed{(\bullet) \text{ is true } \forall\, n \in \mathbb{N}}$

$$\therefore (*) \text{ and } (\bullet) \text{ are both true } \forall\, n \in \mathbb{N} \Rightarrow \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq \sum_{cyc}\frac{a^n + \lambda b^n}{a^{n+1} + \lambda b^{n+1}}$$

$\forall\, a, b, c > 0 \,\big|\, \lambda \geq 0$ and $n \in \mathbb{N}$, " = " iff $a = b = c$ (QED)