ROMANIAN MATHEMATICAL MAGAZINE

If $0 < x, y, z < 1, x + y + z = \frac{3}{2}$, then :

$$\sum_{cyc} \frac{\sqrt{1+x^2}}{1-x} \geq 3\sqrt{5}$$

Proposed by Marin Chirciu-Romania

Solution 1 by Soumava Chakraborty-Kolkata-India, Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} \frac{\sqrt{1+x^2}}{1-x} = \sum_{cyc} \frac{\left(\sqrt[4]{1+x^2}\right)^2}{1-x} \overset{\text{Bergstrom}}{\geq} \frac{\left(\sum_{cyc} \sqrt[4]{1+x^2}\right)^2}{3-(x+y+z)} \overset{x+y+z=\frac{3}{2}}{=}$$

$$\frac{2}{3} \cdot \left(\sum_{cyc} \sqrt[4]{1+x^2}\right)^2 \overset{?}{\geq} 3\sqrt{5} \Leftrightarrow \sum_{cyc} \sqrt[4]{1+x^2} \underset{(*)}{\overset{?}{\geq}} \frac{3\sqrt[4]{5}}{\sqrt{2}}$$

$$\text{Now, } f(t) = \sqrt[4]{1+t^2}\ \forall\, t \in (0,1) \text{ is convex} \because f''(t) = \frac{2-t}{4(1+t^2)^{\frac{7}{4}}} > 0\ (\because 0 < t < 1)$$

$$\therefore \sum_{cyc} \sqrt[4]{1+x^2} \overset{\text{Jensen}}{\geq} 3\sqrt[4]{1+\left(\frac{x+y+z}{3}\right)^2} \overset{x+y+z=\frac{3}{2}}{=} \frac{3\sqrt[4]{5}}{\sqrt{2}} \Rightarrow (*) \text{ is true}$$

$$\therefore \sum_{cyc} \frac{\sqrt{1+x^2}}{1-x} \geq 3\sqrt{5}\ \forall\, x, y, z \in (0,1), '' ='' \text{ iff } x = y = z = \frac{1}{2}\ \text{(QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

We will first prove a lemma that for all $x \in (0,1)$,

$$\frac{\sqrt{1+x^2}}{1-x} \geq \frac{12x-1}{\sqrt{5}}.$$

The inequality is successively equivalent to

$$5(1+x^2) \geq (1-x)^2(12x-1)^2 \Leftrightarrow 2 + 13x - 94x^2 + 156x^3 - 72x^4 \geq 0$$

$$\Leftrightarrow (1-2x)^2[2 + 3x + 18x(1-x)] \geq 0,$$

which is true for all $x \in (0,1)$, with equality iff $x = \frac{1}{2}$.

Using this lemma, we have

$$\frac{\sqrt{1+x^2}}{1-x} + \frac{\sqrt{1+y^2}}{1-y} + \frac{\sqrt{1+z^2}}{1-z} \geq \frac{12(x+y+z)-3}{\sqrt{5}} = 3\sqrt{5},$$

as desired. Equality holds iff $x = y = z = \frac{1}{2}$.