ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ and $\lambda \geq 1$, then :

$$\frac{a}{a+\lambda b}+\frac{b}{b+\lambda c}+\frac{c}{c+\lambda a}+\frac{a^2+b^2+c^2}{4(ab+bc+ca)} \geq \frac{3}{\lambda+1}+\frac{1}{4}$$

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$$\frac{a}{a+\lambda b}+\frac{b}{b+\lambda c}+\frac{c}{c+\lambda a}+\frac{a^2+b^2+c^2}{4(ab+bc+ca)}-\frac{3}{\lambda+1}-\frac{1}{4}$$

$$=\sum_{cyc}\frac{a^2}{a^2+\lambda ab}+\frac{a^2+b^2+c^2}{4(ab+bc+ca)}-\frac{3}{\lambda+1}-\frac{1}{4} \overset{\text{Bergstrom}}{\geq}$$

$$\frac{\sum_{cyc}a^2+2\sum_{cyc}ab}{\sum_{cyc}a^2+\lambda\sum_{cyc}ab}-\frac{3}{\lambda+1}+\frac{\sum_{cyc}a^2}{4\sum_{cyc}ab}-\frac{1}{4}=\frac{x+2y}{x+\lambda y}-\frac{3}{\lambda+1}+\frac{x-y}{4y}$$

$$\left(x=\sum_{cyc}a^2 \text{ and } y=\sum_{cyc}ab\right)=\frac{x+\lambda x+2y+2\lambda y-3x-3\lambda y}{(\lambda+1)(x+\lambda y)}+\frac{x-y}{4y}$$

$$=\frac{\lambda(x-y)-2(x-y)}{(\lambda+1)(x+\lambda y)}+\frac{x-y}{4y}=(x-y)\left(\frac{\lambda-2}{(\lambda+1)(x+\lambda y)}+\frac{1}{4y}\right)$$

$$=\frac{(x-y)(4\lambda y-8y+\lambda x+x+\lambda^2 y+\lambda y)}{4y(\lambda+1)(x+\lambda y)} \overset{\substack{\lambda\geq 1 \\ \text{and} \\ \because x-y\geq 0}}{\geq} \frac{(x-y)(4y-8y+x+x+y+y)}{4y(\lambda+1)(x+\lambda y)}$$

$$=\frac{2(x-y)^2}{4y(\lambda+1)(x+\lambda y)}\geq 0 \therefore \frac{a}{a+\lambda b}+\frac{b}{b+\lambda c}+\frac{c}{c+\lambda a}+\frac{a^2+b^2+c^2}{4(ab+bc+ca)}\geq$$

$$\frac{3}{\lambda+1}+\frac{1}{4}\ \forall\, a,b,c>0 \,\middle|\, \lambda\geq 1, ''='' \text{ iff } a=b=c \text{ (QED)}$$