ROMANIAN MATHEMATICAL MAGAZINE

If $a,\mathbf{b},\mathbf{c}>0$ and $\lambda\geq 1$, then :

$$\frac{a}{a+\lambda b}+\frac{b}{b+\lambda c}+\frac{c}{c+\lambda a}+\frac{9(a^2+b^2+c^2)}{8(a+b+c)^2}\geq\frac{3}{\lambda+1}+\frac{3}{8}$$

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$$\forall\, n\geq\frac{9}{8},\ \frac{a}{a+\lambda b}+\frac{b}{b+\lambda c}+\frac{c}{c+\lambda a}-\frac{3}{\lambda+1}+\frac{n(a^2+b^2+c^2)}{(a+b+c)^2}-\frac{n}{3}$$

$$=\sum_{cyc}\frac{a^2}{a^2+\lambda ab}-\frac{3}{\lambda+1}+n\left(\frac{\sum_{cyc}a^2}{\left(\sum_{cyc}a\right)^2}-\frac{1}{3}\right)$$

$$\overset{\text{Bergstrom}}{\geq}\frac{\sum_{cyc}a^2+2\sum_{cyc}ab}{\sum_{cyc}a^2+\lambda\sum_{cyc}ab}-\frac{3}{\lambda+1}+n\left(\frac{3\sum_{cyc}a^2-\left(\sum_{cyc}a\right)^2}{3\left(\sum_{cyc}a\right)^2}\right)$$

$$\overset{n\geq\frac{9}{8}}{\geq}\frac{x+2y}{x+\lambda y}-\frac{3}{\lambda+1}+\frac{9}{8}\left(\frac{3x-x-2y}{3(x+2y)}\right)\left(x=\sum_{cyc}a^2\ \text{and}\ y=\sum_{cyc}ab\right)$$

$$=\frac{x+\lambda x+2y+2\lambda y-3x-3\lambda y}{(\lambda+1)(x+\lambda y)}+\frac{3(x-y)}{4(x+2y)}=\frac{\lambda(x-y)-2(x-y)}{(\lambda+1)(x+\lambda y)}+\frac{3(x-y)}{4(x+2y)}$$

$$=(x-y)\left(\frac{\lambda-2}{(\lambda+1)(x+\lambda y)}+\frac{3}{4(x+2y)}\right)$$

$$=\frac{(x-y)\big(4(\lambda-2)(x+2y)+3(\lambda+1)(x+\lambda y)\big)}{4(\lambda+1)(x+\lambda y)(x+2y)}$$

$$=\frac{(x-y)\left((3\lambda^2+11\lambda-16)y+7\lambda x-5x\right)}{4(\lambda+1)(x+\lambda y)(x+2y)}$$

$$=\frac{(x-y)\left((3\lambda^2+11\lambda-16)y+2\lambda x+5\lambda x-5x\right)}{4(\lambda+1)(x+\lambda y)(x+2y)}$$

$$\overset{x\geq y}{\geq}\frac{(x-y)\left((3\lambda^2+11\lambda-16)y+2\lambda y+5(\lambda-1)x\right)}{4(\lambda+1)(x+\lambda y)(x+2y)}$$

$$=\frac{(x-y)\left((3\lambda^2+13\lambda-16)y+5(\lambda-1)x\right)}{4(\lambda+1)(x+\lambda y)(x+2y)}\geq 0$$

$$\because\ \lambda-1\geq 0\ \text{and}\ 3\lambda^2+13\lambda-16\overset{\lambda\geq 1}{\geq}0\ \text{and}\ x-y\geq 0$$

$$\therefore\ \frac{a}{a+\lambda b}+\frac{b}{b+\lambda c}+\frac{c}{c+\lambda a}+\frac{n(a^2+b^2+c^2)}{(a+b+c)^2}\geq\frac{n}{3}+\frac{3}{\lambda+1}$$

$$\forall\, a,b,c>0\ \big|\ n\geq\frac{9}{8}\ \text{and}\ \lambda\geq 1\ \text{and putting}\ n=\frac{9}{8},\ \text{we get :}$$

$$\frac{a}{a+\lambda b}+\frac{b}{b+\lambda c}+\frac{c}{c+\lambda a}-\frac{3}{\lambda+1}+\frac{9(a^2+b^2+c^2)}{8(a+b+c)^2}-\frac{3}{8}\geq 0$$

$$\Rightarrow \frac{a}{a+\lambda b}+\frac{b}{b+\lambda c}+\frac{c}{c+\lambda a}+\frac{9(a^2+b^2+c^2)}{8(a+b+c)^2}\geq \frac{3}{\lambda+1}+\frac{3}{8}$$

$$\forall\, a,b,c>0 \,\big|\, \lambda\geq 1,''='' \text{ iff } a=b=c\ (\text{QED})$$