ROMANIAN MATHEMATICAL MAGAZINE

If $a,\mathrm{b},\mathrm{c}>0$ *with* $a\mathrm{bc}=1$ *and* $\lambda\geq 2,\mathrm{n}\geq 0$, then :

$$\sum_{\mathrm{cyc}}\frac{a^3+\lambda}{a^3(\mathrm{b}+\mathrm{nc})}\geq\frac{3(\lambda+1)}{\mathrm{n}+1}$$

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$$\sum_{\mathrm{cyc}}\frac{a^3+\lambda}{a^3(\mathrm{b}+\mathrm{nc})}=\sum_{\mathrm{cyc}}\frac{a}{a\mathrm{b}+\mathrm{nc}a}+\lambda.\sum_{\mathrm{cyc}}\frac{\mathrm{b}^2\mathrm{c}^2}{a^3\mathrm{b}^3\mathrm{c}^2+\mathrm{nc}^3a^3\mathrm{b}^2}\overset{\text{Bergstrom}}{\geq}$$

$$\frac{\left(\sum_{\mathrm{cyc}}\sqrt{a}\right)^2}{(\mathrm{n}+1)\left(\sum_{\mathrm{cyc}}a\mathrm{b}\right)}+\frac{\lambda\left(\sum_{\mathrm{cyc}}a\mathrm{b}\right)^2}{a^2\mathrm{b}^2\mathrm{c}^2\left(\sum_{\mathrm{cyc}}a\mathrm{b}\right)}\overset{a\mathrm{bc}=1}{=}$$

$$\frac{1}{\mathrm{n}+1}.\left(\frac{\sum_{\mathrm{cyc}}a+2\sum_{\mathrm{cyc}}\sqrt{a\mathrm{b}}}{\sum_{\mathrm{cyc}}a\mathrm{b}}+\lambda\sum_{\mathrm{cyc}}a\mathrm{b}\right)$$

$$\geq\frac{1}{\mathrm{n}+1}.\left(\frac{\sqrt{3\sum_{\mathrm{cyc}}a\mathrm{b}}+2\sqrt{a\mathrm{bc}}.\sum_{\mathrm{cyc}}\frac{1^{\frac{3}{2}}}{\sqrt{a}}}{\sum_{\mathrm{cyc}}a\mathrm{b}}+\lambda\sum_{\mathrm{cyc}}a\mathrm{b}\right)\overset{\substack{\text{Radon}\\ \text{and }\because\, a\mathrm{bc}=1}}{\geq}$$

$$\frac{1}{\mathrm{n}+1}.\left(\frac{\sqrt{3\sum_{\mathrm{cyc}}a\mathrm{b}}+2.\frac{\sqrt{27}}{\sqrt{\sum_{\mathrm{cyc}}a}}}{\sum_{\mathrm{cyc}}a\mathrm{b}}+\lambda\sum_{\mathrm{cyc}}a\mathrm{b}\right)\overset{a\mathrm{bc}=1}{=}$$

$$\frac{1}{\mathrm{n}+1}.\left(\frac{\sqrt{3\sum_{\mathrm{cyc}}a\mathrm{b}}+18.\frac{1}{\sqrt{3a\mathrm{bc}\sum_{\mathrm{cyc}}a}}}{\sum_{\mathrm{cyc}}a\mathrm{b}}+\lambda\sum_{\mathrm{cyc}}a\mathrm{b}\right)$$

$$\geq\frac{1}{\mathrm{n}+1}.\left(\frac{\sqrt{3\sum_{\mathrm{cyc}}a\mathrm{b}}+\frac{18}{\sum_{\mathrm{cyc}}a\mathrm{b}}}{\sum_{\mathrm{cyc}}a\mathrm{b}}+\lambda\sum_{\mathrm{cyc}}a\mathrm{b}\right)$$

$$=\frac{1}{\mathrm{n}+1}.\left(\frac{x+\frac{54}{x^2}}{\frac{x^2}{3}}+\lambda.\frac{x^2}{3}\right)\left(x=\sqrt{3\sum_{\mathrm{cyc}}a\mathrm{b}}\overset{\mathrm{A-G}}{\geq}3.\sqrt[6]{a^2\mathrm{b}^2\mathrm{c}^2}\overset{a\mathrm{bc}=1}{=}3\right)$$

$$=\frac{1}{\mathrm{n}+1}.\left(\frac{9x^3+486+\lambda x^6}{3x^4}\right)\overset{?}{\geq}\frac{3(\lambda+1)}{\mathrm{n}+1}\Leftrightarrow\lambda x^4(x^2-9)+9x^3+486-9x^4\underset{(*)}{\overset{?}{\geq}}0$$

$$\text{Now, }x^2-9\geq 0\text{ and }\lambda\geq 0$$

$$\therefore\text{ LHS of }(*)\geq 2x^4(x^2-9)+9x^3+486-9x^4$$

$$=2x^6-27x^4+9x^3+486=$$

$$= (x-3)\Big((x-3)(2x^4+12x^3+27x^2+63x+135)+243\Big) \geq 0 \ (\because x \geq 3)$$

$$\Rightarrow (*) \text{ is true } \therefore \sum_{cyc} \frac{a^3+\lambda}{a^3(b+nc)} \geq \frac{3(\lambda+1)}{n+1}$$

$$\forall\, a,b,c > 0 \,\Big|\, abc = 1 \text{ and } \lambda \geq 2, n \geq 0, '' ='' \text{ iff } a = b = c = 1 \ (\text{QED})$$