ROMANIAN MATHEMATICAL MAGAZINE

If $a, b > 0$ and $n \in \mathbb{N}$ with $n \geq 2$, then :

$$4\left(\frac{a^n}{b+1}+\frac{b^n}{a+1}\right)+(2n-1)\left(\frac{1}{a}+\frac{1}{b}\right) \geq 4n+2$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

WLOG we may assume $a \geq b$ and then : $\frac{a}{b+1} \geq \frac{b}{a+1}$ and

$$a^{n-1} \geq b^{n-1}\ (\because n-1 \geq 1) \therefore 4\left(\frac{a^n}{b+1}+\frac{b^n}{a+1}\right) \overset{\text{Chebyshev}}{\geq}$$

$$2\left(\frac{a}{b+1}+\frac{b}{a+1}\right)(a^{n-1}+b^{n-1})$$

$$\overset{\text{Bernoulli}}{\geq} 2\left(\frac{a}{b+1}+\frac{b}{a+1}\right)\left(1+(a-1)(n-1)+1+(b-1)(n-1)\right)$$

$$= 2\left(\frac{a}{b+1}+\frac{b}{a+1}\right)\left(2+(n-1)(a+b-2)\right)$$

$$= 4\left(\frac{a}{b+1}+\frac{b}{a+1}\right)+2(n-1)(a+b-2)\left(\frac{a}{b+1}+\frac{b}{a+1}\right)$$

$$\Rightarrow 4\left(\frac{a^n}{b+1}+\frac{b^n}{a+1}\right)+(2n-1)\left(\frac{1}{a}+\frac{1}{b}\right)-(4n+2) \geq$$

$$4\left(\frac{a}{b+1}+\frac{b}{a+1}\right)+2(n-1)(a+b-2)\left(\frac{a}{b+1}+\frac{b}{a+1}\right)$$

$$+2(n-1)\left(\frac{1}{a}+\frac{1}{b}\right)+\frac{1}{a}+\frac{1}{b}-4(n-1)-6$$

$$= 2(n-1)\left((a+b-2)\left(\frac{a}{b+1}+\frac{b}{a+1}\right)+\frac{1}{a}+\frac{1}{b}-2\right)$$

$$+4\left(\frac{a}{b+1}+\frac{b}{a+1}\right)+\frac{1}{a}+\frac{1}{b}-6$$

$$\therefore \boxed{\begin{gathered}4\left(\frac{a^n}{b+1}+\frac{b^n}{a+1}\right)+(2n-1)\left(\frac{1}{a}+\frac{1}{b}\right)-(4n+2) \geq \\ 2(n-1)\left((a+b-2)\left(\frac{a}{b+1}+\frac{b}{a+1}\right)+\frac{1}{a}+\frac{1}{b}-2\right)+4\left(\frac{a}{b+1}+\frac{b}{a+1}\right)+\frac{1}{a}+\frac{1}{b}-6\end{gathered}}$$

$$\rightarrow (1)$$

$$\text{Now, } (a+b-2)\left(\frac{a}{b+1}+\frac{b}{a+1}\right)+\frac{1}{a}+\frac{1}{b}-2$$

$$= (a+b-2)\left(\frac{a}{b+1}+\frac{b}{a+1}\right)+\frac{4}{a+b}+\frac{(a-b)^2}{ab(a+b)}-2$$

$$= (a+b-2)\left(\frac{a}{b+1}+\frac{b}{a+1}-\frac{2}{a+b}\right)+\frac{(a-b)^2}{ab(a+b)}$$

$$= (a+b-2)\left(\left(\frac{a}{b+1}-\frac{1}{a+b}\right)+\left(\frac{b}{a+1}-\frac{1}{a+b}\right)\right)+\frac{(a-b)^2}{ab(a+b)}$$

$$= (a+b-2)\left(\frac{a^2+ab-b-1}{(a+b)(b+1)}+\frac{b^2+ab-a-1}{(a+b)(a+1)}\right)+\frac{(a-b)^2}{ab(a+b)}$$

$$= (a+b-2)\left(\frac{(a-1)(a+1)+b(a-1)}{(a+b)(b+1)}+\frac{(b-1)(b+1)+a(b-1)}{(a+b)(a+1)}\right)$$

$$+\frac{(a-b)^2}{ab(a+b)}=\frac{(a+b-2)(a+b+1)}{a+b}.\left(\frac{a-1}{b+1}+\frac{b-1}{a+1}\right)+\frac{(a-b)^2}{ab(a+b)}$$

$$=\frac{(a+b-2)(a+b+1)}{a+b}.\frac{a^2-1+b^2-1}{(b+1)(a+1)}+\frac{(a-b)^2}{ab(a+b)}$$

$$=\frac{(a+b-2)(a+b+1)((a+b)^2+(a-b)^2-4)}{2(a+b)(a+1)(b+1)}+\frac{(a-b)^2}{ab(a+b)}$$

$$=\frac{(a+b-2)(a+b+1)\left((a+b-2)(a+b+2)+(a-b)^2\right)}{2(a+b)(a+1)(b+1)}+\frac{(a-b)^2}{ab(a+b)}$$

$$=\frac{(a+b+1)(a+b+2)(a+b-2)^2}{2(a+b)(a+1)(b+1)}+\frac{(a+b-2)(a+b+1)(a-b)^2}{2(a+b)(a+1)(b+1)}$$

$$+\frac{(a-b)^2}{ab(a+b)}$$

$$=\frac{(a+b+1)(a+b+2)(a+b-2)^2}{2(a+b)(a+1)(b+1)}+\frac{(a-b)^2}{a+b}.\left(\frac{(a+b-2)(a+b+1)}{2(a+1)(b+1)}+\frac{1}{ab}\right)$$

$$\therefore \boxed{\begin{gathered}(a+b-2)\left(\frac{a}{b+1}+\frac{b}{a+1}\right)+\frac{1}{a}+\frac{1}{b}-2=\frac{(a+b+1)(a+b+2)(a+b-2)^2}{2(a+b)(a+1)(b+1)}\\+\frac{(a-b)^2}{2ab(a+b)(a+1)(b+1)}.\left(ab(a+b-2)(a+b+1)+2(a+b+1+ab)\right)\end{gathered}}$$

$$\to (2)$$

Now, $ab(a+b-2)(a+b+1)+2(a+b+1+ab)=$

$$\frac{ab(a+b-2)(a+b+1)}{2}+2(a+b+1)+\frac{ab(a+b-2)(a+b+1)}{2}+2ab$$

$$=\frac{a+b+1}{2}.(ab(a+b)-2ab+4)+\frac{ab}{2}.((a+b)^2-(a+b)-2+4)\overset{A-G}{\geq}$$

$$\frac{a+b+1}{2}.\left(2ab.\sqrt{ab}-2ab+4\right)+\frac{ab}{2}.\left(\left(a+b-\frac{1}{2}\right)^2+\frac{7}{4}\right)$$

$$=(a+b+1)(t^3-t^2+2)+\frac{ab}{2}.\left(\left(a+b-\frac{1}{2}\right)^2+\frac{7}{4}\right)\ (t=\sqrt{ab})$$

$$=(a+b+1)(t+1)((t-1)^2+1)+\frac{ab}{2}.\left(\left(a+b-\frac{1}{2}\right)^2+\frac{7}{4}\right)>0$$

$$\Rightarrow \boxed{ab(a+b-2)(a+b+1)+2(a+b+1+ab)>0}\to (3)\ \therefore (2),(3)\Rightarrow$$

ROMANIAN MATHEMATICAL MAGAZINE

$$(a+b-2)\left(\frac{a}{b+1}+\frac{b}{a+1}\right)+\frac{1}{a}+\frac{1}{b}-2\geq 0\rightarrow (4)$$

$$\therefore (1),(4)\ and\ (n-1)\geq 1\Rightarrow 4\left(\frac{a^n}{b+1}+\frac{b^n}{a+1}\right)+(2n-1)\left(\frac{1}{a}+\frac{1}{b}\right)-(4n+2)$$

$$\geq 2\left((a+b-2)\left(\frac{a}{b+1}+\frac{b}{a+1}\right)+\frac{1}{a}+\frac{1}{b}-2\right)+4\left(\frac{a}{b+1}+\frac{b}{a+1}\right)+\frac{1}{a}+\frac{1}{b}-6$$

$$=2(a+b)\left(\frac{a}{b+1}+\frac{b}{a+1}\right)-4\left(\frac{a}{b+1}+\frac{b}{a+1}\right)+3\left(\frac{1}{a}+\frac{1}{b}\right)-10$$

$$+4\left(\frac{a}{b+1}+\frac{b}{a+1}\right)\overset{\text{Bergstrom}}{\geq}$$

$$2(a+b)\left(\frac{(a+b)^2}{2ab+a+b}\right)+\frac{12}{a+b}-10\geq 2(a+b)\left(\frac{(a+b)^2}{\frac{(a+b)^2}{2}+a+b}\right)+\frac{12}{a+b}-10$$

$$\overset{?}{\geq} 0\Leftrightarrow 2m.\frac{m^2}{\frac{m^2}{2}+m}+\frac{12}{m}-10\overset{?}{\geq} 0\ (m=a+b)\Leftrightarrow 2m^3-5m^2-4m-12\overset{?}{\geq} 0$$

$$\Leftrightarrow (2m+3)(m-2)^2\geq 0\rightarrow \text{true}$$

$$\therefore 4\left(\frac{a^n}{b+1}+\frac{b^n}{a+1}\right)+(2n-1)\left(\frac{1}{a}+\frac{1}{b}\right)-(4n+2)\geq 0$$

$$\Rightarrow 4\left(\frac{a^n}{b+1}+\frac{b^n}{a+1}\right)+(2n-1)\left(\frac{1}{a}+\frac{1}{b}\right)\geq 4n+2$$

$\forall\ a,b>0\ and\ n\in\mathbb{N}$ with $n\geq 2$,″ =″ iff $a=b=1$ (QED)