ROMANIAN MATHEMATICAL MAGAZINE

If $a, \mathrm{b}, \mathrm{c} > 0$ and $a + \mathrm{b} + \mathrm{c} = 3$, then :

$$\sum_{cyc} \frac{\mathrm{bc}}{\sqrt[4]{2a^4 + 14}} \le \frac{3}{2}$$

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$$\frac{1}{\sqrt[4]{2a^4+14}} \overset{?}{\le} \frac{9-a}{16} \Leftrightarrow \frac{1}{2a^4+14} \overset{?}{\le} \frac{(9-a)^4}{65536}$$

$$(\because a, \mathrm{b}, \mathrm{c} > 0 \wedge a + \mathrm{b} + \mathrm{c} = 3 \Rightarrow 9 > 3 > a) \Leftrightarrow (2a^4+14)(9-a)^4 \overset{?}{\ge} 65536$$

$$\Leftrightarrow a^8 - 36a^7 + 486a^6 - 2916a^5 + 6568a^4 - 252a^3$$

$$+3402a^2 - 20412a + 13159 \overset{?}{\ge} 0$$

$$\Leftrightarrow (a-1)^2\left(a^6 - 34a^5 + 417a^4 - 2048a^3 + 2055a^2 + 5906a + 13159\right) \overset{?}{\ge} 0$$

$$\Leftrightarrow (a-1)^2((a-8)^2(a^4 - 18a^3 + 65a^2 + 144a + 199) + 423 - 126a) \overset{?}{\ge} 0$$

$$\Leftrightarrow (a-1)^2\left((a-8)^2(a^2(a-5)(a-13) + 144a + 199) + 45 + 126(3-a)\right) \overset{?}{\ge} 0$$

$$\to \text{true} \because 0 < a < 3 \Rightarrow (a-5), (a-13) < 0 \Rightarrow a^2(a-5)(a-13) > 0 \text{ and}$$

$$126(3-a) > 0 \therefore \frac{1}{\sqrt[4]{2a^4+14}} \le \frac{9-a}{16} \text{ and analogs}$$

$$\Rightarrow \sum_{cyc} \frac{\mathrm{bc}}{\sqrt[4]{2a^4+14}} \le \sum_{cyc} \frac{\mathrm{bc}(9-a)}{16} \overset{?}{\le} \frac{3}{2} \Leftrightarrow 3\sum_{cyc} a\mathrm{b} - a\mathrm{bc} \overset{?}{\le} 8$$

$$\overset{a+\mathrm{b}+\mathrm{c}=3}{\Leftrightarrow} \left(\sum_{cyc} a\right)\left(\sum_{cyc} a\mathrm{b}\right) - a\mathrm{bc} \overset{?}{\le} \frac{8}{27}\left(\sum_{cyc} a\right)^3$$

$$\Leftrightarrow 8\sum_{cyc} a^3 \underset{(*)}{\overset{?}{\ge}} 6a\mathrm{bc} + 3\sum_{cyc} a^2\mathrm{b} + 3\sum_{cyc} a\mathrm{b}^2$$

$$\text{Now, Schur + AM - GM} \Rightarrow 6\sum_{cyc} a^3 \ge 3\sum_{cyc} a^2\mathrm{b} + 3\sum_{cyc} a\mathrm{b}^2 \to (1) \text{ and AM - GM} \Rightarrow$$

$$2\sum_{cyc} a^3 \ge 6a\mathrm{bc} \to (2) \therefore (1) + (2) \Rightarrow (*) \text{ is true}$$

$$\therefore \sum_{cyc} \frac{\mathrm{bc}}{\sqrt[4]{2a^4+14}} \le \frac{3}{2} \ \forall\, a, \mathrm{b}, \mathrm{c} > 0 \,\middle|\, a + \mathrm{b} + \mathrm{c} = 3, \text{''}=\text{''} \text{ iff } a = \mathrm{b} = 1 \text{ (QED)}$$