ROMANIAN MATHEMATICAL MAGAZINE

If $0 < x, y, z < 1, x + y + z = \frac{3}{2}$ *and* $0 \le \lambda \le 1$, then :

$$\sum_{cyc} \frac{\sqrt{\lambda + x^2}}{1 - x} \ge 3.\sqrt{4\lambda + 1}$$

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$$\sqrt{4\lambda+1} - \frac{\sqrt{\lambda+x^2}}{1-x} = \frac{4\lambda+1-\frac{\lambda+x^2}{(1-x)^2}}{\sqrt{4\lambda+1}+\frac{\sqrt{\lambda+x^2}}{1-x}} = \frac{\frac{(4\lambda+1)(1+x^2-2x)-\lambda-x^2}{(1-x)^2}}{\sqrt{4\lambda+1}+\frac{\sqrt{\lambda+x^2}}{1-x}}$$

$$= \frac{1-2x+\lambda(4x^2-8x+3)}{\sqrt{4\lambda+1}.(1-x)^2+(1-x).\sqrt{\lambda+x^2}} = \frac{1-2x-\lambda(1-2x)(2x-3)}{\sqrt{4\lambda+1}.(1-x)^2+(1-x).\sqrt{\lambda+x^2}}$$

$$= \frac{(1-2x)(1-\lambda(2x-3))}{\sqrt{4\lambda+1}.(1-x)^2+(1-x).\sqrt{\lambda+x^2}} \therefore \boxed{\sqrt{4\lambda+1} - \frac{\sqrt{\lambda+x^2}}{1-x} - \frac{(1-2x)(4\lambda+2)}{\sqrt{4\lambda+1}}}$$

$$= (1-2x)\left(\frac{1-\lambda(2x-3)}{\sqrt{4\lambda+1}.(1-x)^2+(1-x).\sqrt{\lambda+x^2}} - \frac{4\lambda+2}{\sqrt{4\lambda+1}}\right)$$

$$= -(1-2x).\frac{(4\lambda+2).\sqrt{4\lambda+1}.(1-2x+x^2)+(4\lambda+2)(1-x).\sqrt{\lambda+x^2} - \sqrt{4\lambda+1}+\lambda(2x-3).\sqrt{4\lambda+1}}{\sqrt{4\lambda+1}.\left(\sqrt{4\lambda+1}.(1-x)^2+(1-x).\sqrt{\lambda+x^2}\right)}$$

$$= -\frac{(1-2x)\left(\begin{array}{c}(4\lambda+2).\sqrt{4\lambda+1}.(1-2x)+(4\lambda+2).\sqrt{4\lambda+1}.x^2+\\(4\lambda+2)\left(\frac{1}{2}-x+\frac{1}{2}\right).\sqrt{\lambda+x^2}-\sqrt{4\lambda+1}+\lambda(2x-1).\sqrt{4\lambda+1}-2\lambda.\sqrt{4\lambda+1}\end{array}\right)}{\sqrt{4\lambda+1}.\left(\sqrt{4\lambda+1}.(1-x)^2+(1-x).\sqrt{\lambda+x^2}\right)}$$

$$= -\frac{(1-2x)\left(\begin{array}{c}(1-2x)(2\lambda+1)\left(2.\sqrt{4\lambda+1}+\sqrt{\lambda+x^2}-2\lambda.\frac{\sqrt{4\lambda+1}}{4\lambda+2}\right)+\\(2\lambda+1)\left(2\left(x^2-\frac{1}{4}+\frac{1}{4}\right).\sqrt{4\lambda+1}+\sqrt{\lambda+x^2}-\sqrt{4\lambda+1}\right)\end{array}\right)}{\sqrt{4\lambda+1}.\left(\sqrt{4\lambda+1}.(1-x)^2+(1-x).\sqrt{\lambda+x^2}\right)}$$

$$= -\frac{(1-2x)\left(\begin{array}{c}(1-2x)(2\lambda+1)\left(2.\sqrt{4\lambda+1}+\sqrt{\lambda+x^2}-2\lambda.\frac{\sqrt{4\lambda+1}}{4\lambda+2}\right)+\\(2\lambda+1)\left(-2\left(\frac{1}{4}-x^2\right).\sqrt{4\lambda+1}+\sqrt{\lambda+x^2}-\frac{1}{2}\sqrt{4\lambda+1}\right)\end{array}\right)}{\sqrt{4\lambda+1}.\left(\sqrt{4\lambda+1}.(1-x)^2+(1-x).\sqrt{\lambda+x^2}\right)}$$

$$= -\frac{(1-2x)\left(\begin{array}{c}(1-2x)(2\lambda+1)\left(2.\sqrt{4\lambda+1}+\sqrt{\lambda+x^2}-2\lambda.\frac{\sqrt{4\lambda+1}}{4\lambda+2}\right)+\\(2\lambda+1)\left(-2\left(\frac{1}{4}-x^2\right).\sqrt{4\lambda+1}-\frac{\frac{1}{4}(4\lambda+1)-\lambda-x^2}{\frac{1}{2}\sqrt{4\lambda+1}+\sqrt{\lambda+x^2}}\right)\end{array}\right)}{\sqrt{4\lambda+1}.\left(\sqrt{4\lambda+1}.(1-x)^2+(1-x).\sqrt{\lambda+x^2}\right)}$$

$$= -\frac{(1-2x)\left(\begin{array}{l}(1-2x)(2\lambda+1)\left(2.\sqrt{4\lambda+1}+\sqrt{\lambda+x^2}-2\lambda.\frac{\sqrt{4\lambda+1}}{4\lambda+2}\right)+\\(2\lambda+1)(1-2x)\left(-\left(\frac{1}{2}+x\right).\sqrt{4\lambda+1}-\frac{\frac{1}{2}+x}{\sqrt{4\lambda+1}+2\sqrt{\lambda+x^2}}\right)\end{array}\right)}{\sqrt{4\lambda+1}.\left(\sqrt{4\lambda+1}.(1-x)^2+(1-x).\sqrt{\lambda+x^2}\right)} =$$

$$-\frac{(1-2x)^2(2\lambda+1)\left(2.\sqrt{4\lambda+1}+\sqrt{\lambda+x^2}-2\lambda.\frac{\sqrt{4\lambda+1}}{4\lambda+2}-\left(\frac{1}{2}+x\right)\left(\sqrt{4\lambda+1}+\frac{1}{\sqrt{4\lambda+1}+2\sqrt{\lambda+x^2}}\right)\right)}{\sqrt{4\lambda+1}.\left(\sqrt{4\lambda+1}.(1-x)^2+(1-x).\sqrt{\lambda+x^2}\right)}$$

$$\le -\frac{(1-2x)^2(2\lambda+1)\left(2.\sqrt{4\lambda+1}+\sqrt{\lambda+x^2}-2\lambda.\frac{\sqrt{4\lambda+1}}{4\lambda+1}-\left(\frac{1}{2}+x\right).\frac{4\lambda+2+2\sqrt{4\lambda+1}.\sqrt{\lambda+x^2}}{\sqrt{4\lambda+1}+2\sqrt{\lambda+x^2}}\right)}{\sqrt{4\lambda+1}.\left(\sqrt{4\lambda+1}.(1-x)^2+(1-x).\sqrt{\lambda+x^2}\right)}$$

$$\boxed{\le} -\frac{(1-2x)^2(2\lambda+1)\left(\left(2.\sqrt{4\lambda+1}+\sqrt{\lambda+x^2}-\left(\frac{1}{2}+x\right).\frac{4\lambda+2+2\sqrt{4\lambda+1}.\sqrt{\lambda+x^2}}{\sqrt{4\lambda+1}+2\sqrt{\lambda+x^2}}\right)-\frac{2\lambda}{\sqrt{4\lambda+1}}\right)}{\sqrt{4\lambda+1}.\left(\sqrt{4\lambda+1}.(1-x)^2+(1-x).\sqrt{\lambda+x^2}\right)}$$

$$= -\frac{(1-2x)^2(2\lambda+1)\left(\frac{8\lambda+1+2x^2-4x\lambda-2x+(4-2x).\sqrt{(4\lambda+1)(\lambda+x^2)}}{\sqrt{4\lambda+1}+2\sqrt{\lambda+x^2}}-\frac{2\lambda}{\sqrt{4\lambda+1}}\right)}{\sqrt{4\lambda+1}.\left(\sqrt{4\lambda+1}.(1-x)^2+(1-x).\sqrt{\lambda+x^2}\right)}$$

$$= -\frac{(1-2x)^2(2\lambda+1)}{\sqrt{4\lambda+1}.\left(\sqrt{4\lambda+1}.(1-x)^2+(1-x).\sqrt{\lambda+x^2}\right)} *$$

$$\left(\frac{4\lambda(2-x)+\left(1+2x^2-2x\right)+(4-2x).\sqrt{(4\lambda+1)(\lambda+x^2)}}{\sqrt{4\lambda+1}+2\sqrt{\lambda+x^2}}-\frac{2\lambda}{\sqrt{4\lambda+1}}\right)$$

$$= -\frac{(1-2x)^2(2\lambda+1)}{\sqrt{4\lambda+1}.\left(\sqrt{4\lambda+1}.(1-x)^2+(1-x).\sqrt{\lambda+x^2}\right)} *$$

$$\left(\frac{\begin{array}{c}4\lambda.\sqrt{4\lambda+1}.(2-x)+\left(1+2x^2-2x\right).\sqrt{4\lambda+1}+(4-2x)(4\lambda+1).\sqrt{\lambda+x^2}-\\2\lambda.\sqrt{4\lambda+1}-4\lambda.\sqrt{\lambda+x^2}\end{array}}{\sqrt{4\lambda+1}.\left(\sqrt{4\lambda+1}+2\sqrt{\lambda+x^2}\right)}\right)$$

$$= -\frac{(1-2x)^2(2\lambda+1)}{\sqrt{4\lambda+1}.\left(\sqrt{4\lambda+1}.(1-x)^2+(1-x).\sqrt{\lambda+x^2}\right)} *$$

$$\left(\frac{2\lambda.\sqrt{4\lambda+1}.(3-2x)+\sqrt{\lambda+x^2}.(4\lambda(3-2x)+4-2x)+(1-x)^2.\sqrt{4\lambda+1}+x^2.\sqrt{4\lambda+1}}{\sqrt{4\lambda+1}.\left(\sqrt{4\lambda+1}+2\sqrt{\lambda+x^2}\right)}\right)$$

$\boxed{\le 0}$ $\because$ $0<x<1$ and $\lambda\ge 0 \Rightarrow 2\lambda.\sqrt{4\lambda+1}.(3-2x)+$
$\sqrt{\lambda+x^2}.(4\lambda(3-2x)+4-2x)+(1-x)^2.\sqrt{4\lambda+1}+x^2.\sqrt{4\lambda+1}>0$

$$\therefore \sqrt{4\lambda+1}-\frac{\sqrt{\lambda+x^2}}{1-x}-\frac{(1-2x)(4\lambda+2)}{\sqrt{4\lambda+1}} \le 0\ \forall\, x\in(0,1) \text{ and } \forall\, \lambda\ge 0$$

$$\Rightarrow \boxed{\frac{\sqrt{\lambda + x^2}}{1 - x} \geq \sqrt{4\lambda + 1} - \frac{(1 - 2x)(4\lambda + 2)}{\sqrt{4\lambda + 1}} \ \forall\, x \in (0,1) \ and \ \forall\, \lambda \geq 0} \ and\ analogs$$

$$\Rightarrow \sum_{\text{cyc}} \frac{\sqrt{\lambda + x^2}}{1 - x} \geq 3.\sqrt{4\lambda + 1} - \frac{4\lambda + 2}{\sqrt{4\lambda + 1}}.\left(3 - 2\sum_{\text{cyc}} x\right)$$

$$\overset{x+y+z=\frac{3}{2}}{=} 3.\sqrt{4\lambda + 1} \ \forall\, x, y, z \in (0,1) \ and \ \forall\, \lambda \in [0,1], \text{''} = \text{''} \ \text{iff } x = y = z = \frac{1}{2} \ \text{(QED)}$$