ROMANIAN MATHEMATICAL MAGAZINE

If $a, b > 0, a + b = 2$ and $0 \le \lambda \le 1$, then :

$$\frac{1}{a^3+\lambda}+\frac{1}{b^3+\lambda}\ge\frac{2}{\lambda+1}$$

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$$\frac{1}{a^3+\lambda}+\frac{1}{b^3+\lambda}\ge\frac{2}{\lambda+1}\Leftrightarrow\left(\frac{1}{a^3+\lambda}-\frac{1}{\lambda+1}\right)+\left(\frac{1}{b^3+\lambda}-\frac{1}{\lambda+1}\right)\ge 0$$

$$\Leftrightarrow\frac{1-a^3}{a^3+\lambda}+\frac{1-b^3}{b^3+\lambda}\ge 0\ \left(\because\frac{1}{\lambda+1}>0\ as\ 0\le\lambda\le 1\right)$$

$$\Leftrightarrow\frac{b^3-a^3b^3+\lambda-\lambda a^3+a^3-a^3b^3+\lambda-\lambda b^3}{(a^3+\lambda)(b^3+\lambda)}\ge 0$$

$$\Leftrightarrow a^3+b^3-2a^3b^3+2\lambda-\lambda(a^3+b^3)\overset{(*)}{\ge}0$$

$$\text{Now, }2=a+b\overset{A-G}{\ge}2.\sqrt{ab}\Rightarrow ab\le 1\Rightarrow a^2b^2\le 1\Rightarrow 2a^3b^3\le 2ab$$
$$\Rightarrow \text{LHS of }(*)\ge a^3+b^3-2ab+2\lambda-\lambda(a^3+b^3)$$
$$\overset{a+b=2}{=}2(a^2-ab+b^2)-2ab+\frac{2\lambda}{4}(a+b)^2-2\lambda(a^2-ab+b^2)$$
$$=2(a-b)^2+\frac{\lambda}{2}(a^2+2ab+b^2-4a^2+4ab-4b^2)=2(a-b)^2-\frac{3\lambda}{2}.(a-b)^2$$
$$=\frac{(a-b)^2}{2}(4-3\lambda)\ge 0\because 4-3\lambda=1+3(1-\lambda)\overset{0\le\lambda\le 1}{\ge}1>0$$
$$\therefore\frac{1}{a^3+\lambda}+\frac{1}{b^3+\lambda}\ge\frac{2}{\lambda+1}\ \forall\, a,b>0\,\big|\,a+b=2\wedge 0\le\lambda\le 1,$$
$$''='' \text{ iff } a=b=1\ (\text{QED})$$