ROMANIAN MATHEMATICAL MAGAZINE

$$If\ a,b,c>0,\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=2\ then:$$

$$\sum\frac{1}{\sqrt{7a^2+ab+b^2}}\le\frac{2}{3}$$

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Solution by Tapas Das-India

$$7a^2+ab+b^2=6a^2+(a^2+b^2)+ab\overset{AM-GM}{\ge}$$
$$\ge 6a^2+2ab+ab=6a^2+3ab=3a(2a+b)\ \ (1)$$

$$\sum\frac{1}{\sqrt{7a^2+ab+b^2}}\overset{(1)}{\le}\sum\frac{1}{\sqrt{3a(2a+b)}}=$$

$$=\sum\sqrt{\frac{1}{3a}\cdot\frac{1}{2a+b}}\overset{AM-GM}{\le}\sum\frac{1}{2}\left(\frac{1}{3a}+\frac{1}{2a+b}\right)=\frac{1}{6}\sum\frac{1}{a}+\frac{1}{2}\sum\frac{1}{2a+b}=$$

$$=\frac{1}{6}\sum\frac{1}{a}+\frac{1}{2}\sum\frac{1}{a+a+b}\overset{AM-HM}{\le}\frac{1}{6}\sum\frac{1}{a}+\frac{1}{2}\cdot\frac{1}{9}\sum\left(\frac{1}{a}+\frac{1}{a}+\frac{1}{b}\right)=$$

$$=\frac{1}{6}.2+\frac{1}{18}(2+2+2)\ \left(as\ \frac{1}{a}+\frac{1}{b}+\frac{1}{c}=2\right)=\frac{1}{3}+\frac{1}{3}=\frac{2}{3}$$

$$Equality\ holds\ \ for\ a=b=c=\frac{3}{2}$$